

On Digraphs with Non-derogatory Adjacency Matrix

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Abstract. Let G be a digraph with n vertices and $A(G)$ be its adjacency matrix. A monic polynomial $f(x)$ of degree at most n is called an annihilating polynomial of G if $f(A(G)) = 0$. G is said to be annihilatingly unique if it possesses a unique annihilating polynomial. We prove that two families of digraphs, i.e., the ladder digraphs and the difans, are annihilatingly unique by studying the similarity invariants of their adjacency matrices respectively.

1. Introduction

All graphs under consideration in this paper are directed, connected, finite, loopless and without multiple arcs. By a digraph $G = (V, E)$, we mean a finite set V (the elements of which are called vertices) together with a set E of ordered pairs of elements of V (these pairs are called arcs). Two vertices are said to be adjacent if they are connected by an arc. Let G be a digraph with n vertices and $A(G)$ be its adjacency matrix. A monic polynomial $f(x)$ of degree at most n is called an annihilating polynomial of G if $f(A(G)) = 0$. Undefined terms and notations can be found in [2] and [3].

Theorem 1. [2] *The annihilating polynomial $f(x)$ of any digraph G with adjacency matrix $A(G)$ is unique if and only if $A(G)$ is non-derogatory, that is, $m(x) = \psi(x)$ where $m(x)$ is the minimum polynomial of $A(G)$ and $\psi(x)$ is the characteristic polynomial of $A(G)$.*

In [4], Lam and Lim gave an explicit expression for the characteristic polynomial of the ladder digraph and showed that the ladder digraph is annihilatingly unique. In this paper, we introduce another approach to show that the ladder digraph and certain digraphs are annihilatingly unique without evaluating their characteristic polynomials.

Definition 1. [1] *For any arbitrary $n \times n$ matrix A , form the characteristic matrix $\lambda I_n - A$ and let $d_j(\lambda)$ denote the g.c.d. (greatest common divisor) of all minors of order j of $\lambda I_n - A$, $j = 1, 2, \dots, n$. These polynomials are called the determinantal divisors*

of $\lambda I_n - A$ and it follows that the quotients $i_j(\lambda) = \frac{d_j(\lambda)}{d_{j-1}(\lambda)}$ for $j = 1, 2, \dots, n$ ($d_0 \equiv 1$) are also polynomials, called the similarity invariants of A .

Theorem 2. [1] *A matrix A is non-derogatory if and only if its first $n-1$ similarity invariants are unity.*

2. The difan AF_n of order n with alternative spokes

Let us investigate a class of digraphs defined as the difan with alternative spokes (arcs). We denote this class of difans by AF_n .

Definition 2. AF_n is a digraph of order $n \geq 3$, consisting of a dipath P_{n-1} (with vertices labelled as $2, 3, \dots, n$) and an additional vertex 1. There is one arc from vertex 1 to vertex $2i$ for $i = 1, 2, \dots$, where $2i \leq n$ and there is one arc from vertex $2i + 1$ to vertex 1 for $i = 1, 2, \dots$, where $2i + 1 \leq n$.

As an example, AF_4 is as shown in Figure 1.

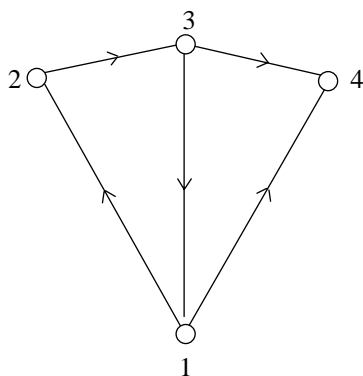


Figure 1. AF_4

The adjacency matrix of AF_4 is given by

$$A(AF_4) = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The characteristic matrix of AF_4 is given by

$$\lambda I_4 - A(AF_4) = \begin{bmatrix} \lambda & -1 & 0 & -1 \\ 0 & \lambda & -1 & 0 \\ -1 & 0 & \lambda & -1 \\ 0 & 0 & 0 & \lambda \end{bmatrix}.$$

It is easily seen that $i_1(\lambda) = 1$ and $i_2(\lambda) = 1$. For $i_3(\lambda)$, the minors of a_{11} and a_{41} of $\lambda I_4 - A(AF_4)$ of order 3 are

$$M_{11} = \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 0 & 0 & \lambda \end{vmatrix} = \lambda^3, \quad M_{41} = \begin{vmatrix} -1 & 0 & -1 \\ \lambda & -1 & 0 \\ 0 & \lambda & -1 \end{vmatrix} = -(1 + \lambda^2).$$

Hence we have $d_3(\lambda) = (\lambda^3, -(1 + \lambda^2)) = 1$ and $i_3(\lambda) = 1$. Thus $A(AF_4)$ is non-derogatory by Theorem 2, and it follows that AF_4 is annihilatingly unique.

In what follows, we shall prove that AF_n is annihilatingly unique for all $n \geq 3$.

Lemma 1. *Let M_{ij} be the minor of $\lambda I_n - A(AF_n)$ at the (i, j) -entry, then $M_{1n} = \lambda^{n-1}$, and*

$$M_{n1} = \begin{cases} 1 + \lambda^2 + \cdots + \lambda^{n-3} & \text{if } n \geq 3 \text{ is odd,} \\ -(1 + \lambda^2 + \cdots + \lambda^{n-4} + \lambda^{n-2}) & \text{if } n \geq 4 \text{ is even.} \end{cases}$$

Proof. We now assume that $n = 2m + 1$ is odd. The characteristic matrix $\lambda I_n - A(AF_n)$ of AF_n is of the form

$$\begin{bmatrix} \lambda & -1 & 0 & -1 & 0 & -1 & \dots & -1 & 0 & -1 & 0 \\ 0 & \lambda & -1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ -1 & 0 & \lambda & -1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda & -1 & 0 & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \dots & \dots & 0 & \lambda & -1 & 0 & 0 \\ -1 & 0 & 0 & \dots & \dots & \dots & \dots & 0 & \lambda & -1 & 0 \\ 0 & 0 & 0 & \dots & \dots & \dots & \dots & \dots & 0 & \lambda & -1 \\ -1 & 0 & 0 & \dots & \dots & \dots & \dots & \dots & \dots & 0 & \lambda \end{bmatrix}$$

and M_{11} is the determinant of an upper triangular matrix with all its entries in the main diagonal equal to λ , hence $M_{11} = \lambda^{n-1}$.

To find the minor M_{n1} , let B be the matrix obtained from $\lambda I_n - A(AF_n)$ by deleting its first column and n -th row. We then evaluate $M_{n1} = \det(B)$ along the first row $(-1, 0, -1, 0, \dots, -1, 0)$ of B . Let further C_{2i+1} be obtained from B by deleting its first row and $(2i + 1)$ -st column, $0 \leq i \leq m - 1$. Then C_{2i+1} is of the form

$$\begin{array}{c} \uparrow \\ 2i \\ \downarrow \\ \uparrow \\ n-2-2i \\ \downarrow \end{array} \left[\begin{array}{ccc|ccc} \lambda & -1 & & & & \\ & \lambda & -1 & 0 & & \\ & & \ddots & \ddots & & \\ 0 & & & \lambda & -1 & \\ & & & & \lambda & \\ \hline & & & & -1 & \\ 0 & & & & \lambda & -1 & 0 \\ & & & & & \lambda & \ddots \\ & & & & & 0 & \ddots & -1 \\ & & & & & & & \lambda & -1 \end{array} \right]$$

hence $\det(C_{2i+1}) = (-1)^{n-2-2i} \lambda^{2i}$. It follows that

$$\begin{aligned} M_{n1} = \det(B) &= \sum_{i=0}^{m-1} (-1)^{1+(2i+1)} (-1) \det(C_{2i+1}) \\ &= 1 + \lambda^2 + \lambda^4 + \dots + \lambda^{n-3} \end{aligned}$$

as required. The case for even n can be treated similarly.

From the characteristic matrix $\lambda I_n - A(AF_n)$, $i_j(\lambda) = 1$ for $j = 1, 2, \dots, n - 2$ irrespective of whether n is odd or even. By Lemma 1, we have shown that $i_{n-1}(\lambda) = 1$. Hence, we obtain the following corollary.

Corollary 1. $A(AF_n)$ with $n \geq 3$ has all its first $n - 1$ similarity invariants equal to unity.

From Lemma 1 and Corollary 1, we obtain the following results.

Theorem 3. $A(AF_n)$ with $n \geq 3$ is non-derogatory.

Corollary 2. AF_n with $n \geq 3$ is annihilatingly unique.

3. The ladder digraph L_{2k} of order $2k$

The annihilating uniqueness of the ladder digraph L_{2k} was proved by Lam and Lim [4]. In this section, we give a shorter proof to their result by using the method introduced in this paper.

Definition 3. The ladder digraph, denoted by L_{2k} (k being a positive integer) is a finite, loopless connected digraph with vertex set $V = \{1, 2, \dots, 2k\}$ together with the following arcs:

$$(i, i + 1), \quad i = 1, 2, \dots, 2k - 1, \quad (2j, 2j - 3), \quad j = 2, 3, \dots, k.$$

The ladder digraph L_{2k} is illustrated in Figure 2.

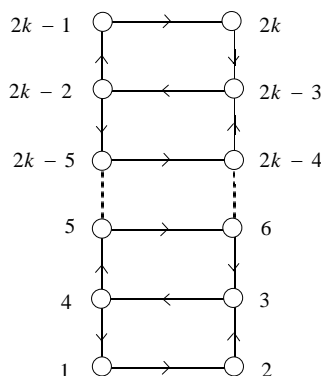


Figure 2. L_{2k}

Theorem 4. L_{2k} is annihilatingly unique.

In general, we have the following result.

Theorem 5. *If the adjacency matrix of a digraph G of order n contains a non-singular lower (upper) triangular submatrix of order $n - 1$, then $A(G)$ is non-derogatory.*

Proof. This is a consequence of Theorem 2.

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