

IDEMPOTENT-SEPARATING EXTENSIONS OF REGULAR SEMIGROUPS

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Received 1 July 2004 and in revised form 10 March 2005

For a regular biordered set E , the notion of E -diagram and the associated regular semigroup was introduced in our previous paper (1995). Given a regular biordered set E , an E -diagram in a category C is a collection of objects, indexed by the elements of E and morphisms of C satisfying certain compatibility conditions. With such an E -diagram A we associate a regular semigroup $\text{Reg}_E(\mathbf{A})$ having E as its biordered set of idempotents. This regular semigroup is analogous to automorphism group of a group. This paper provides an application of $\text{Reg}_E(\mathbf{A})$ to the idempotent-separating extensions of regular semigroups. We introduced the concept of crossed pair and used it to describe all extensions of a regular semigroup S by a group E -diagram A . In this paper, the necessary and sufficient condition for the existence of an extension of S by A is provided. Also we study cohomology and obstruction theories and find a relationship with extension theory for regular semigroups.

1. Introduction

If $\pi : T \rightarrow S$ is an idempotent-separating surjective homomorphism of regular semigroups, then the kernel of π defines a group $E(S)$ -diagram $\mathbf{A} : \underline{C}(E(S)) \rightarrow \mathbf{GR}$ that factors through $\mathbf{D}(B(E(S)))$ and π induces an idempotent-separating homomorphism $\Psi : S \rightarrow (\text{Reg}_{E(S)}(\mathbf{A}))/\text{Inn}_{E(S)}(\mathbf{A})$ ((T, π) is called an *extension of S by the group $E(S)$ -diagram \mathbf{A} with abstract kernel Ψ*). In this paper, we discuss the following extension problem for regular semigroups.

Given $\Psi : S \rightarrow (\text{Reg}_{E(S)}(\mathbf{A}))/\text{Inn}_{E(S)}(\mathbf{A})$, find all extensions of S by A with abstract kernel Ψ . Of course, given Ψ , is it possible that no extension of S by A with abstract kernel Ψ can exist. In this connection, an *obstruction theory* is developed for finding extensions of S by A which induce the given Ψ .

In Section 1, we introduce the concept of a *crossed pair* and use it to describe all extensions of S by a group $E(S)$ -diagram \mathbf{A} . In Section 2, we associate with each $\Psi : S \rightarrow \text{Reg}_{E(S)}(\mathbf{A})/\text{Inn}_{E(S)}(\mathbf{A})$ a three-dimensional cohomology class in the Leech cohomology of S^I . We show in Theorem 3.6 that the vanishing of this cohomology class is necessary and sufficient condition for the existence of an extension of S by \mathbf{A} with abstract kernel Ψ .

We further show that if Ψ has an extension, then the set of all equivalence classes of extensions of S by \mathbf{A} with abstract kernel Ψ is in bijective correspondence with the set of all elements of certain second cohomology group.

Before proceeding further, let us recall some known definitions and results.

For any regular semigroup S , we denote by $E(S)$ the set of idempotents of S and by $V(x)$ the set of inverses of an element $x \in S$. Thus $V(x) = \{x' \in S : xx'x = x, x'xx' = x'\}$. A pair of elements (x, x') such that $x' \in V(x)$ is called a *regular pair* in S .

A homomorphism $\theta : T \rightarrow S$ of regular semigroups is called *idempotent-separating* if θ is one-to-one on the idempotents of T . A *congruence* ρ is called idempotent-separating if the associated projection homomorphism is idempotent-separating. Let ρ be an idempotent-separating congruence on S . Then $\rho \subseteq \mathbf{H}$. For each $e \in E(S)$, let $\mathbf{K}_e = \rho(e) = \{x \in S : xpe\}$. Then \mathbf{K}_e is a subgroup of the maximal subgroup \mathbf{H}_e of S . The family $\mathbf{K} = \{\mathbf{K}_e : e \in E(S)\}$, where $(\mathbf{K}_S)_e = \{a \in \mathbf{H}_e : af = fa \text{ for each idempotent } f \leq e\}$.

Definition 1.1 [7]. Let S be a regular semigroup. For each $e \in E(S)$, let \mathbf{K}_e be a subgroup of \mathbf{H}_e . Then $\mathbf{K} = \{\mathbf{K}_e : e \in E(S)\}$ is called a *group kernel normal system* of S if it satisfies

- (i) $af = fa$ for all $a \in \mathbf{K}_e$ and for all $f \in E(S)$ such that $f \leq e$,
- (ii) $x'\mathbf{K}_{xx'}x \subseteq \mathbf{K}_{x'x}$ for each regular pair (x, x') of S .

PROPOSITION 1.2 [3]. *Let S be a regular semigroup. Let $\mathbf{K} = \{\mathbf{K}_e : e \in E(S)\}$ be a group kernel normal system of S . Define*

$$\begin{aligned} \rho_{\mathbf{K}} = \{ & (x, y) \in S \times S : \text{for some } x' \in V(x) \text{ and } y' \in V(y), \\ & xx' = yy', x'x = y'y, \text{ and } y'x \in \mathbf{K}_{x'x}\}. \end{aligned} \tag{1.1}$$

Then $\rho_{\mathbf{K}}$ is an idempotent-separating congruence on S whose kernel is the group kernel normal system \mathbf{K} of S . Conversely, if ρ is an idempotent-separating congruence on S , then the kernel \mathbf{K} of ρ is a group kernel normal system of S and $\rho_{\mathbf{K}} = \rho$.

Let us recall some results from [10, 13].

Let E be a regular biordered set. We write $\omega^r = \{(e, f) : fe = e\}$, $\omega^l = \{(e, f) : ef = e\}$ and $\mathbf{R} = \omega^r \cap (\omega^r)^{-1}$, $\mathbf{L} = \omega^l \cap (\omega^l)^{-1}$, $\omega = \omega^r \cap \omega^l$.

Definition 1.3. Let E be a regular biordered set and $\mathbf{G}(E)$ the ordered groupoid of E -chains of E [13]. The category $\underline{\mathbf{C}}(E)$ has objects as the elements of E and a morphism from e to f is a pair (e, c) , where $c = c(e_0, \dots, e_n) \in \mathbf{G}(E)$ such that $e \geq e_0$ and $e_n = f$.

If $(e, c) : e \rightarrow f$, $(f, c') : f \rightarrow g$ are two morphisms, with $c = c(e_0, \dots, e_n)$, $c' = c(f_0, \dots, f_m)$, then the composite is given by $(e, c)(f, c') = (e, (c * f_0)c')$, where $(c * f_0)c'$ is the composite of $(c * f_0)$ and c' in $\mathbf{G}(E)$. The identity morphism at e is $(e, c(e))$ and the associativity of the composition follows from the transitivity property of the ordered groupoid $\mathbf{G}(E)$ [13, Proposition 3.3].

Throughout this paper, S will denote a regular semigroup with biordered set of idempotents $E = E(S)$. $B(E)$ the universal regular idempotent generated semigroup on E [13]. $\mathbf{C}(S)$ denotes the category with the set of idempotents $E(S)$ as its objects and morphism from an object e to an object f is a triple $(e, x, x') : e \rightarrow f$, where (x, x') is a regular pair

such that $e \geq xx'$ and $x'x = f$. Composition of morphisms is defined by $(e, x, x')(x'x, y, y') = (e, xy, y'x')$. Define an equivalence relation \sim on the morphisms of $\mathbf{C}(S)$ as follows: if $(e, x, x'), (e, y, y') : e \rightarrow f$ are two morphisms, then $(e, x, x') \sim (e, y, y')$ if and only if there exist idempotents $e_0, e_1, \dots, e_n \in \omega(e)$ with $(e_{i-1}, e_i) \in \mathbf{R} \cup \mathbf{L}$, $i = 1$ to n , such that $e_0 = xx', e_n = yy'$, and $(y, y') = (e_n e_{n-1} \cdots e_0 x, x' e_0 e_1 \cdots e_n)$. Then $\mathbf{D}(S) = \mathbf{C}(S)/\sim$ is the quotient category of $\mathbf{C}(S)$. If we view the underlying groupoid of the inductive groupoid $\mathbf{G}(S)$ of S as a subcategory of $\mathbf{C}(S)$ via the embedding $(x, x') \rightarrow (xx', x, x')$, then the evaluation map $\varepsilon_S : c(e_0, e_1, \dots, e_n) \rightarrow (e_0 e_1 \cdots e_n, e_n e_{n-1} \cdots e_0) : \mathbf{G}(E) \rightarrow \mathbf{G}(S)$ extends to a functor $\bar{\varepsilon}_S : \underline{\mathbf{C}}(E) \rightarrow \mathbf{C}(S)$ such that $\bar{\varepsilon}_S(e, c) \rightarrow (e, \bar{\varepsilon}_S(c))$ for every morphism (e, c) of $\underline{\mathbf{C}}(E)$. In particular, by taking $S = B(E)$ we obtain a functor $\bar{\varepsilon}_{B(E)} : \underline{\mathbf{C}}(E) \rightarrow \mathbf{C}(B(E))$. By [13, Theorem 6.9], the inclusion $E \subseteq S$ extends uniquely to an idempotent-separating homomorphism $\delta : B(E) \rightarrow S$. If $\mathbf{C}(\delta) : (e, x, x') \rightarrow (e, x\delta, x'\delta) : \mathbf{C}(B(E)) \rightarrow \mathbf{C}(S)$ is the induced functor, then $\mathbf{C}(\delta)\bar{\varepsilon}_{B(E)} = \bar{\varepsilon}_S$.

If $\theta : S \rightarrow S'$ is a homomorphism of regular semigroups, then the maps $e \rightarrow e\theta; [e, x, x'] \rightarrow [e\theta, x\theta, x'\theta]$ define a functor $\mathbf{D}(\theta) : \mathbf{D}(S) \rightarrow \mathbf{D}(S')$. Let E be a biordered set. For each $e \in E$, the inclusion $\omega(e) \subseteq E$ induces a functor $\mathbf{D}(B(\omega(e))) \rightarrow \mathbf{D}(B(E))$. Let $[e, x, x'], [e, y, y'] : e \rightarrow f$ be two morphisms of $\mathbf{D}(B(E))$, then

$$[e, x, x'] = [e, y, y'] \quad \text{if } x = y \text{ or } x' = y'. \tag{1.2}$$

LEMMA 1.4. *Let $[e, x, x'] : e \rightarrow g$ be any morphism of $\mathbf{D}(B(\omega(e)))$ with domain e . Then $[e, x, x'] = [e, g, g]$ in $\mathbf{D}(B(\omega(e)))$ and hence in $\mathbf{D}(B(E))$.*

Let S be a regular semigroup. Let ρ_S be the maximum idempotent-separating congruence on S . Then the kernel \mathbf{K}_S of ρ_S defines a group-valued functor $\mathbf{K}_S : \mathbf{C}(S) \rightarrow \mathbf{GR}$, where \mathbf{GR} denotes the category of groups, which associates to each object e of $\mathbf{C}(S)$ the group $(\mathbf{K}_S)_e$ and to each morphism $(e, x, x') : e \rightarrow f$ the group homomorphism $\mathbf{K}_S(e, x, x') : (\mathbf{K}_S)_e \rightarrow (\mathbf{K}_S)_f$ given by $(a)\mathbf{K}_S(e, x, x') = x'ax$.

PROPOSITION 1.5. *If ρ is an idempotent-separating congruence on S , then $K^\rho : \mathbf{C}(S) \rightarrow \mathbf{GR}$ defined by $K^\rho(e) = \rho(e); K^\rho(e, x, x') = \mathbf{K}_S(e, x, x')/K^\rho(e)$ is a subfunctor of \mathbf{K}_S . Conversely, if $\mathbf{K}' : \mathbf{C}(S) \rightarrow \mathbf{GR}$ is a subfunctor of \mathbf{K}_S , then $\mathbf{K}' = \{\mathbf{K}'_e : e \in E(S)\}$ is a group kernel normal system of S and defines, by (1.1), an idempotent-separating congruence $\rho_{\mathbf{K}'}$ on S . Further $\rho \rightarrow K^\rho$ defines a bijective correspondence between the idempotent-separating congruences on S and the subfunctors of \mathbf{K}_S .*

Let $\pi : T \rightarrow S$ be an idempotent-separating homomorphism from T onto S . Then $\mathbf{K}^{\pi\pi^{-1}} : \mathbf{C}(T) \rightarrow \mathbf{GR}$ factors through $\mathbf{D}(T)$. That is, there is a functor $\text{Ker } \pi : \mathbf{D}(T) \rightarrow \mathbf{GR}$ such that the diagram

$$\begin{array}{ccc}
 \mathbf{C}(T) & \xrightarrow{\mathbf{K}^{\pi\pi^{-1}}} & \mathbf{GR} \\
 & \searrow & \nearrow \text{Ker } \pi \\
 & \mathbf{D}(T) &
 \end{array}
 \tag{1.3}$$

is commutative. Thus $(\text{Ker } \pi)_e = \{a \in T : a\pi = e\pi\}$, $e \in E(T)$, and $a\text{Ker } \pi(e, x, x') = x'ax$, $a \in (\text{Ker } \pi)_e$.

LEMMA 1.6 [9, Lemma 4.1]. *Let $\pi : T \rightarrow S$ be an idempotent-separating onto homomorphism of regular semigroups. If $t\pi = u\pi = x$, $t, u \in T$, $x \in S$, then for each $e \in E(S) \cap \mathbf{L}_x$, there exists a unique element $a \in T$ such that $u = ta$ and $a\pi = e$.*

Definition 1.7. Let \mathbf{GR} be the category of groups. By an E -diagram in \mathbf{GR} we mean a functor $\mathbf{A} : \underline{\mathbf{C}}(E) \rightarrow \mathbf{GR}$ which factors through $\mathbf{C}(B(E))$. In other words, a functor $\mathbf{A} : \underline{\mathbf{C}}(E) \rightarrow \mathbf{GR}$ is an E -diagram in \mathbf{GR} if there is a (necessarily unique) functor $\hat{\mathbf{A}} : \mathbf{C}(B(E)) \rightarrow \mathbf{GR}$ such that $\mathbf{A} = \hat{\mathbf{A}}\bar{\varepsilon}_{B(E)}$.

Observe that if \mathbf{A} is an E -diagram in \mathbf{GR} , then for any two morphisms $(e, c), (e, c') : e \rightarrow f$ in $\underline{\mathbf{C}}(E)$, $\mathbf{A}(e, c) = \mathbf{A}(e, c')$ whenever $\bar{\varepsilon}_{B(E)}(c) = \bar{\varepsilon}_{B(E)}(c')$.

Let \mathbf{A} be a contravariant E -diagram in \mathbf{GR} . Then there exists a contravariant functor $\hat{\mathbf{A}} : \mathbf{C}(B(E)) \rightarrow \mathbf{GR}$ such that $\mathbf{A} = \hat{\mathbf{A}}\bar{\varepsilon}_{B(E)}$. For each $e \in E$, let \mathbf{A}^e denote the composite

$$\mathbf{A}^e : \underline{\mathbf{C}}(\omega(e)) \xrightarrow{i_e} \underline{\mathbf{C}}(E) \xrightarrow{\mathbf{A}} \mathbf{GR}. \tag{1.4}$$

Define $\mathbf{G}(\mathbf{A})$ to be the category whose objects are the elements of E . A morphism $e \rightarrow f$ is a pair of (α, ϕ) consisting of an ω -isomorphism $\alpha : \omega(e) \rightarrow \omega(f)$ and a natural isomorphism $\phi : \mathbf{A}^e \rightarrow \mathbf{A}^f \underline{\mathbf{C}}(\alpha)$, where $\underline{\mathbf{C}}(\alpha) : \underline{\mathbf{C}}(\omega(e)) \rightarrow \underline{\mathbf{C}}(\omega(f))$ is the functor defined by the ω -isomorphism α , and $\mathbf{A}^f \underline{\mathbf{C}}(\alpha)$ is the composite

$$\mathbf{C}(\omega(e)) \xrightarrow{\underline{\mathbf{C}}(\alpha)} \underline{\mathbf{C}}(\omega(f)) \xrightarrow{\mathbf{A}^f} \mathbf{GR}. \tag{1.5}$$

Note that the natural isomorphism ϕ assigns to each object h in $\underline{\mathbf{C}}(\omega(e))$ an isomorphism $\phi_h : \mathbf{A}_h \rightarrow \mathbf{A}_{(h)\alpha}$ such that, for any morphism $(h, c) : h \rightarrow k$ in $\underline{\mathbf{C}}(\omega(e))$, the following diagram commutes:

$$\begin{array}{ccc} \mathbf{A}_h & \xrightarrow{\phi_h} & \mathbf{A}_{(h)\alpha} \\ \mathbf{A}(h,c) \uparrow & & \uparrow \mathbf{A}(h\alpha, c\alpha) \\ \mathbf{A}_k & \xrightarrow{\phi_k} & \mathbf{A}_{(k)\alpha} \end{array} \tag{1.6}$$

The composite of two morphisms $(\alpha, \phi) : e \rightarrow f$, $(\beta, \Psi) : f \rightarrow g$ is given by $(\alpha, \phi)(\beta, \Psi) = (\alpha\beta, \phi(\underline{\mathbf{C}}(\alpha)\Psi))$, where $\alpha\beta$ is the composite $\omega(e) \xrightarrow{\alpha} \omega(f) \xrightarrow{\beta} \omega(g)$ and the natural isomorphism $\phi(\underline{\mathbf{C}}(\alpha)\Psi) : \mathbf{A}^e \rightarrow \mathbf{A}^g \underline{\mathbf{C}}(\alpha\beta)$ is defined by

$$(\phi(\underline{\mathbf{C}}(\alpha)\Psi))_h = \phi_h \circ \Psi_{(h)\alpha} : \mathbf{A}_h \rightarrow \mathbf{A}_{(h)\alpha\beta} \tag{1.7}$$

for all $h \in \omega(e)$. For each object e , $(1_e, \mathbf{1}_e) : e \rightarrow e$, where $1_e : \omega(e) \rightarrow \omega(e)$ is the identity ω -isomorphism and $\mathbf{1}_e : \mathbf{A}^e \rightarrow \mathbf{A}^e$ is the identity isomorphism, is the identity morphism of e . For an E -chain $c = c(e_0, e_1, \dots, e_n) \in \mathbf{G}(E)$, define $\varepsilon : \mathbf{G}(E) \rightarrow \mathbf{G}(\mathbf{A})$ by $\varepsilon(c) = (\alpha^c, \phi^c)$,

where $\alpha^c : \omega(e_0) \rightarrow \omega(e_n)$ and $\phi^c : \mathbf{A}^{e_0} \rightarrow \mathbf{A}^{e_n} \underline{C}(\alpha^c)$ are such that

$$(h)\alpha^c = (h)\tau_E(c) = \tau(e_0, e_1)\tau(e_1, e_2) \cdots \tau(e_{n-1}, e_n), \tag{1.8}$$

$\tau_E : \mathbf{G}(E) \rightarrow T^*(E)$ is the evaluation map of the inductive groupoid $T^*(E)$ of ω -isomorphism of E , and $\phi_h^c = \mathbf{A}(h, h * c)^{-1} : \mathbf{A}_h \rightarrow \mathbf{A}_{(h)\alpha^c}$ for every $h \in \omega(e_0)$. By [10], it follows that $(\mathbf{G}(\mathbf{A}), \varepsilon)$ is an inductive groupoid.

Let $\mathbf{A} : \underline{C}(E) \rightarrow \mathbf{GR}$ be an E -diagram in \mathbf{GR} . Let $\text{Reg}_E(\mathbf{A})$ be the quotient of $\mathbf{G}(\mathbf{A})$ by the equivalence relation ρ , where for any two morphisms $(\alpha, \phi) : e \rightarrow f, (\beta, \Psi) : f \rightarrow g$ in $\mathbf{G}(\mathbf{A})$,

$$(\alpha, \phi)\rho(\beta, \Psi) \iff e\mathbf{R}g, \quad f\mathbf{L}h, \varepsilon(c(e, g))(\beta, \Psi) = (\alpha, \phi)\varepsilon(c(f, h)). \tag{1.9}$$

Also if $[\alpha, \phi], [\beta, \Psi]$ are the elements of $\text{Reg}_E(\mathbf{A})$ with representatives $(\alpha, \phi) : e \rightarrow f, (\beta, \Psi) : f \rightarrow g$, then as in [13]

$$[\alpha, \phi][\beta, \Psi] = [(\alpha, \phi) \circ_1 (\beta, \Psi)] = [((\alpha, \phi) * fl)(\alpha^c, \phi^c)(lg * (\beta, \Psi))], \tag{1.10}$$

where $l \in S(f, g)$, the sandwich set of f and g , and $c = c(fl, l, lg)$. Also note that $E(\text{Reg}_E(\mathbf{A})) = [1_e, 1_e]$.

LEMMA 1.8. *Let $([x], [y])$ be a regular pair in S such that $[x][y] = [1_e]$ and $[y][x] = [1_f]$. Then there exists $z : e \rightarrow f$ in the inductive groupoid \mathbf{G} such that $[z] = [x]$ and $[z^{-1}] = [y]$.*

2. Idempotent-separating extensions of regular semigroups

Consider a regular semigroup T and an idempotent-separating homomorphism $\pi : T \rightarrow S$ of T onto S . Let $\text{Ker } \pi : \mathbf{D}(T) \rightarrow \mathbf{GR}$ be the group-valued functor defined by the kernel of π . The inverse $i = (\pi/E(T))^{-1} : E \rightarrow E(T)$ of the biorder isomorphism $\pi/E(T) : E(T) \rightarrow E(S) = E$ extends to an idempotent-separating homomorphism $\hat{\mathbf{i}} : B(E) \rightarrow T$ by [13, Theorem 6.9] and hence induces a functor $\mathbf{D}(\hat{\mathbf{i}}) : \mathbf{D}(B(E)) \rightarrow \mathbf{D}(T)$. If $\pi_1 : \underline{C}(E) \rightarrow \mathbf{D}(B(E))$ denotes the functor

$$(e, c(e_0, \dots, e_n)) \longrightarrow [e, e_0e_1 \cdots e_n, e_n e_{n-1} \cdots e_0], \tag{2.1}$$

then the composite

$$\mathbf{A}^\pi = \text{Ker } \pi \mathbf{D}(\hat{\mathbf{i}}) \pi_1 : \underline{C}(E) \longrightarrow \mathbf{GR} \tag{2.2}$$

is a group E -diagram which factors through $\mathbf{D}(B(E))$. Thus $\mathbf{A}_e^\pi = \{t \in T : t\pi = e\}$ for each object e of $\underline{C}(E)$ and

$$\mathbf{A}^\pi(e, c(e_0, \dots, e_n)) = \text{Ker } \pi[ei, (e_0i) \cdots (e_ni), (e_ni) \cdots (e_0i)] : \mathbf{A}_e \longrightarrow \mathbf{A}_f \tag{2.3}$$

for each morphism $(e, c(e_0, \dots, e_n)) : e \rightarrow f$ of $\underline{C}(E)$. This observation motivates the following.

Definition 2.1. Let $\mathbf{A} : \underline{C}(E) \rightarrow \mathbf{GR}$ be a (covariant) group E -diagram that factors through $\mathbf{D}(B(E))$. An *extension of the regular semigroup S by the group E -diagram \mathbf{A}* is a triple $\varepsilon_T = (T, \pi, U)$ consisting of a regular semigroup T , an idempotent-separating homomorphism $\pi : T \rightarrow S$ of T onto S , and a natural isomorphism of functors $U : \mathbf{A} \rightarrow \mathbf{A}^\pi$.

Remark 2.2. Let $e \in E$ and let \mathbf{A}^e be the composite $\underline{C}(\omega(e)) \xrightarrow{i_e} \underline{C}(E) \xrightarrow{\mathbf{A}} \mathbf{GR}$. For each $x \in \mathbf{A}_e$, we define a natural isomorphism $\eta^x : \mathbf{A}^e \rightarrow \mathbf{A}^e$ as follows. Given $h \in \omega(e)$, let $x_h = (x)\mathbf{A}(e, h) \in \mathbf{A}_h$, and let $\eta_h^x : \mathbf{A}_h \rightarrow \mathbf{A}_h$; $(a)\eta_h^x = x_h^{-1}ax_h$ be the inner automorphism defined by x_h . If $m = (h, c(h_0, h_1, \dots, h_n)) : h \rightarrow k$ is a morphism of $\underline{C}(\omega(e))$, then

$$x_h \mathbf{A}(m) = (x)\mathbf{A}(e, h)\mathbf{A}(m) = (x)\mathbf{A}((e, h)m) = (x)\mathbf{A}(e, k) = x_k \tag{2.4}$$

and therefore the diagram

$$\begin{array}{ccc} \mathbf{A}_h & \xrightarrow{\eta_h^x} & \mathbf{A}_h \\ \mathbf{A}(m) \downarrow & & \downarrow \mathbf{A}(m) \\ \mathbf{A}_k & \xrightarrow{\eta_k^x} & \mathbf{A}_k \end{array} \tag{2.5}$$

is commutative. Thus the map $h \rightarrow \eta_h^x$, $h \in \omega(e)$, defines a natural isomorphism $\eta^x : \mathbf{A}^e \rightarrow \mathbf{A}^e$. If $\text{Reg}_E(\mathbf{A})$ is the regular semigroups of partial isomorphisms of the E -diagram \mathbf{A} , then $[1_e, \eta^x] \in \text{Reg}_E(\mathbf{A})$, where $1_e : \omega(e) \rightarrow \omega(e)$ is the identity isomorphism. Clearly $\eta^x \eta^y = \eta^{xy}$ for all $x, y \in \mathbf{A}_e$ and hence the map

$$\eta : x \rightarrow [1_e, \eta^x] : \mathbf{A}_e \longrightarrow \text{Reg}_E(\mathbf{A}) \tag{2.6}$$

is a homomorphism. Denote the image of \mathbf{A}_e under η by $\text{Inn}(\mathbf{A})_e$. Then $\text{Inn}(\mathbf{A})_e$ is a subgroup of the maximal group $\mathbf{H}_{[1_e, 1_e]}$ of $\text{Reg}_E(\mathbf{A})$. We write

$$\text{Inn}_E(\mathbf{A}) = \{ \text{Inn}_E(\mathbf{A}) \}_{e \in E}. \tag{2.7}$$

PROPOSITION 2.3. $\text{Inn}_E(\mathbf{A})$ is a group kernel normal system in $\text{Reg}_E(\mathbf{A})$.

Proof. Let $x \in \mathbf{A}_e$ and $h \in \omega(e)$. Then $[1_e, \eta^x][1_h, \mathbf{1}_h] = [1_h, \eta^x_h] = [1_h, \mathbf{1}_h][1_e, \eta^x]$. Next, let (s, s') be a regular pair in $\text{Reg}_E(\mathbf{A})$ such that $ss' = [1_e, \mathbf{1}_e]$ and $s's = [1_f, \mathbf{1}_f]$. Using Lemma 1.8, choose a morphism $(\alpha, \phi) : e \rightarrow f$ in $G(\mathbf{A})$ such that $[\alpha, \phi] = s$, $[\alpha^{-1}, \phi^{-1}] = s'$. Then, for any $x \in \mathbf{A}_e$, we have $s'(x)\eta s = [\alpha^{-1}, \phi^{-1}][1_e, \eta^x][\alpha, \phi] = [1_f, \eta^{(x)\phi_e}] = ((x)\phi_e)\eta \in \text{Inn}(\mathbf{A})_f$. Hence, by Definition 1.1, $\text{Inn}_E(\mathbf{A})$ is a group kernel normal system in $\text{Reg}_E(\mathbf{A})$. □

Let $\text{Reg}_E(\mathbf{A})/\text{Inn}_E(\mathbf{A})$ be the quotient of $\text{Reg}_E(\mathbf{A})$ by the idempotent-separating congruence determined by $\text{Inn}_E(\mathbf{A})$ (see Proposition 1.2) and let

$$t : \text{Reg}_E(\mathbf{A}) \longrightarrow \text{Reg}_E(\mathbf{A})/\text{Inn}_E(\mathbf{A}) \tag{2.8}$$

be the associated projection homomorphism.

We next define the centre of the E -diagram \mathbf{A} . For any $e \in E$, let

$$\begin{aligned} \mathbf{Z}(\mathbf{A})_e &= \text{Ker} \{ \eta : \mathbf{A}_e \rightarrow \text{Reg}_E(\mathbf{A}) \} = \{ a \in \mathbf{A}_e : (a)\eta = [1_e, \mathbf{1}_e] \} \\ &= \{ a \in \mathbf{A}_e : (a)A(e, h) = a_h \in \mathbf{Z}(\mathbf{A}_h) \text{ for every } h \in \omega(e) \}. \end{aligned} \tag{2.9}$$

Evidently $\mathbf{Z}(\mathbf{A})_e$ is an abelian normal subgroup of \mathbf{A}_e . If $(e, c = c(e_0, \dots, e_n)) : e \rightarrow f$ is a morphism in $\underline{C}(E)$, then $(\mathbf{Z}(\mathbf{A})_e)\mathbf{A}(e, c) \subseteq \mathbf{Z}(\mathbf{A})_f$. For, if $a \in \mathbf{Z}(\mathbf{A})_e$, then for any element $h \in \omega(f)$, letting $c' = c * h = c(h_0, h_1, \dots, h_n)$, we have $(a)\mathbf{A}(e, c)\mathbf{A}(f, h) = (a)\mathbf{A}((e, c)(f, h)) = (a)\mathbf{A}(e, c * h) = (a)\mathbf{A}(e, c') = (a)\mathbf{A}(e, h_0)\mathbf{A}(h_0, c') \in \mathbf{Z}(\mathbf{A})_h$, since $(a)\mathbf{A}(e, h_0) \in \mathbf{Z}(\mathbf{A})_{h_0}$ and $\mathbf{A}(h_0, c') : \mathbf{A}_{h_0} \rightarrow \mathbf{A}_h$ is an isomorphism of groups. Therefore, the maps

$$e \rightarrow \mathbf{Z}(\mathbf{A})_e; \quad (e, c) \rightarrow \mathbf{A}(e, c) \mid \mathbf{Z}(\mathbf{A})_e : \mathbf{Z}(\mathbf{A})_e \rightarrow \mathbf{Z}(\mathbf{A})_f \tag{2.10}$$

define a functor $\mathbf{Z}(\mathbf{A}) : \underline{C}(E) \rightarrow \mathbf{GR}$, which is a subfunctor of $\mathbf{A} : \underline{C}(E) \rightarrow \mathbf{GR}$. Since $\mathbf{Z}(\mathbf{A})_e$'s are abelian groups, we may also view $\mathbf{Z}(\mathbf{A})$ as a functor from $\underline{C}(E)$ to \mathbf{Ab} , the category of abelian groups.

Definition 2.4. The functor $\mathbf{Z}(\mathbf{A}) : \underline{C}(E) \rightarrow \mathbf{Ab}$ is called the *centre* of \mathbf{A} .

PROPOSITION 2.5. *The sequence*

$$1 \rightarrow \mathbf{Z}(\mathbf{A}) \xrightarrow{i} \mathbf{A} \xrightarrow{\eta} \text{Reg}_E(\mathbf{A}) \xrightarrow{t} \text{Reg}_E(\mathbf{A})/\text{Inn}_E(\mathbf{A}) \rightarrow 1 \tag{2.11}$$

is exact in the sense that t is an idempotent-separating onto homomorphism and the sequence

$$1 \rightarrow \mathbf{Z}(\mathbf{A})_e \xrightarrow{ie} \mathbf{A}_e \rightarrow (\text{Ker } t)_e \rightarrow 1 \tag{2.12}$$

is an exact sequence of groups for each $e \in E$.

Since \mathbf{A} factors through $\mathbf{D}(B(E))$, so is the centre $\mathbf{Z}(\mathbf{A}) : \underline{C}(E) \rightarrow \mathbf{Ab}$. Let $\text{Reg}_E(\mathbf{Z}(\mathbf{A}))$ be the regular semigroup of partial isomorphisms of $\mathbf{Z}(\mathbf{A})$. If $(\alpha, \phi) : e \rightarrow f$ is a morphism in $\mathbf{G}(\mathbf{A})$, then for each $h \in \omega(e)$, $\phi_h : \mathbf{A}_h \rightarrow \mathbf{A}_{(h)\alpha}$ induces by restriction an isomorphism $\bar{\phi}_h : \mathbf{Z}(\mathbf{A})_h \rightarrow \mathbf{Z}(\mathbf{A})_{(h)\alpha}$ and therefore the map $h \rightarrow \bar{\phi}_h$ defines a natural $\bar{\phi} : \mathbf{Z}(\mathbf{A})^e \rightarrow \mathbf{Z}(\mathbf{A})^f \underline{C}(\alpha)$. Thus we have an idempotent-separating homomorphism

$$u : \text{Reg}_E(\mathbf{A}) \rightarrow \text{Reg}_E \mathbf{Z}(\mathbf{A}) \tag{2.13}$$

defined by $[\alpha, \phi]u = [\alpha, \bar{\phi}]$, for $[\alpha, \phi] \in \text{Reg}_E(\mathbf{A})$. If $x \in \mathbf{A}_e$, then clearly $\bar{\eta}^x : \mathbf{Z}(\mathbf{A})^e \rightarrow \mathbf{Z}(\mathbf{A})^e$ is the identity natural isomorphism. Hence, u induces an idempotent-separating homomorphism

$$v : \text{Reg}_E(\mathbf{A})/\text{Inn}_E(\mathbf{A}) \rightarrow \text{Reg}_E \mathbf{Z}(\mathbf{A}) \tag{2.14}$$

such that $tv = u$.

Definition 2.6. Two extensions $\varepsilon_T = (T, \pi, U)$ and $\varepsilon_{T'} = (T', \pi', U')$ of S by \mathbf{A} are equivalent if there exists an isomorphism $\theta : T \rightarrow T'$ of regular semigroups such that

- (i) $\theta\pi' = \pi$,
- (ii) for each $e \in E$, the diagram

$$\begin{array}{ccc}
 & \mathbf{A}_e & \\
 U_e \swarrow & & \searrow U'_e \\
 \mathbf{A}_e^\pi & \xrightarrow{\theta} & \mathbf{A}_e^\pi
 \end{array} \tag{2.15}$$

is commutative.

This defines an equivalence relation on any set of extensions of S by \mathbf{A} .

Given an extension $\varepsilon_T = (T, \pi, U)$ of S by \mathbf{A} , we usually identify \mathbf{A} with \mathbf{A}^π so that $U = \mathbf{1}$, the identity natural isomorphism on \mathbf{A} .

Let $\varepsilon_T = (T, \pi, \mathbf{1})$ be an extension of S by \mathbf{A} and let $\text{Reg}_E(\mathbf{A})$ be the regular semigroup of partial isomorphisms of \mathbf{A} . We define a map $\bar{\mu} : T \rightarrow \text{Reg}_E(\mathbf{A})$ as follows. Given $x \in T$, choose $x' \in V(x)$ and let

$$(x)\bar{\mu} = [\beta(x, x'), \Psi(x, x')], \tag{2.16}$$

where the ω -isomorphism $\beta(x, x') : \omega((xx')\pi) \rightarrow \omega((x'x)\pi)$ is given by

$$(h)\beta(x, x') = (x'\pi)h(x\pi), \quad h \in \omega((xx')\pi), \tag{2.17}$$

and the natural isomorphism $\Psi(x, x') : \mathbf{A}^{(xx')\pi} \rightarrow A^{(x'x)\pi} \underline{C}(\beta(x, x'))$ sends each object h of $\underline{C}(\omega(xx')\pi)$ to the isomorphism

$$\Psi_h(x, x') : a \rightarrow x'ax : \mathbf{A}_h \rightarrow \mathbf{A}_{(x'\pi)h(x\pi)}. \tag{2.18}$$

The element $(x)\bar{\mu}$ is independent of the chosen $x' \in V(x)$, and $x \rightarrow (x)\bar{\mu}$ defines an idempotent-separating homomorphism $\bar{\mu} : T \rightarrow \text{Reg}_E(\mathbf{A})$ such that $(e)\bar{\mu} = [1_{e\pi}, 1_{e\pi}]$ for every $e \in E(T)$. These facts are immediate from [10, Theorem 1.6], as $\bar{\mu}$ is essentially the idempotent-separating homomorphism induced by the composite: $\text{Ker } \pi : \mathbf{D}(T) \rightarrow \mathbf{GR}$ with the projection functor $\mathbf{C}(T) \rightarrow \mathbf{D}(T) : (e, x, x') \rightarrow [e, x, x']$. If $x \in \mathbf{A}_e$ with inverse x^{-1} in $\mathbf{A}_{e'}$, then from (2.17) and (2.18) we obtain

$$(x)\bar{\mu} = [\beta(x, x^{-1}), \Psi(x, x^{-1})] = [1_e, \eta^x] \in (\text{Inn } \mathbf{A})_e, \tag{2.19}$$

where $\eta : \mathbf{A}_e \rightarrow \text{Reg}_E(\mathbf{A})$ is as in (2.6). Hence, $\bar{\mu}$ induces an idempotent-separating homomorphism $\Psi : S \rightarrow \text{Reg}_E(\mathbf{A})/\text{Inn}_E(\mathbf{A})$ completing the square

$$\begin{array}{ccc}
 T & \xrightarrow{\bar{\mu}} & \text{Reg}_E(\mathbf{A}) \\
 \pi \downarrow & & \downarrow t \\
 S & \xrightarrow{\Psi} & \text{Reg}_E(\mathbf{A})/\text{Inn}_E(\mathbf{A})
 \end{array} \tag{2.20}$$

where t is the projection homomorphism. From (2.17) it is clear that the diagram

$$\begin{array}{ccc}
 S & \xrightarrow{\Psi} & \text{Reg}_E(\mathbf{A})/\text{Inn}_E(\mathbf{A}) \\
 \searrow^{\theta_S} & & \swarrow_{\theta'_A} \\
 & T_E &
 \end{array}
 \tag{2.21}$$

is commutative. Here as in [13], θ_S denotes the fundamental representation of S and θ'_A is the idempotent-separating homomorphism induced by the fundamental representation $\theta_A : \text{Reg}_E(\mathbf{A}) \rightarrow T_E : [\alpha, \phi] \mapsto [\alpha]$ of $\text{Reg}_E(\mathbf{A})$.

Definition 2.7. Let $\mathbf{A} : \underline{C}(E) \rightarrow \mathbf{GR}$ be a group E -diagram that factors through $\mathbf{D}(B(E))$. Let $\Psi : S \rightarrow \text{Reg}_E(\mathbf{A})/\text{Inn}_E(\mathbf{A})$ be an idempotent-separating homomorphism such that diagram (2.21) is commutative. Then the triple (S, Ψ, \mathbf{A}) , or just Ψ , is called an *abstract kernel*.

The discussion preceding Definition 2.7 shows that an extension $\varepsilon_T = (T, \pi, \mathbf{1})$ of S by \mathbf{A} defines an abstract kernel $\Psi : S \rightarrow \text{Reg}_E(\mathbf{A})/\text{Inn}_E(\mathbf{A})$ which we call *the abstract kernel of the extension ε_T* . ε_T is called *an extension of the abstract kernel Ψ* .

Remark 2.8. If $\Psi : S \rightarrow \text{Reg}_E(\mathbf{A})/\text{Inn}_E(\mathbf{A})$ is an abstract kernel, then the following two properties of Ψ are immediate from the commutativity of diagram (2.21):

- (i) $[1_e, \mathbf{1}_e] \in (e)\Psi$ for every $e \in E$,
- (ii) if $[\alpha, \phi] \in (x)\Psi$, then for some $x' \in V(x)$, there is a representative $(\alpha', \phi') : xx' \rightarrow x'x$ of $[\alpha, \phi]$ in the inductive groupoid $\mathbf{G}(\mathbf{A})$ such that $(e)\alpha' = x'ex$ for all $e \in \omega(xx')$.

Note that if (ii) holds for one $x' \in V(x)$, then it holds for all $x' \in V(x)$.

The rest of the section is devoted to a description of extensions of S by \mathbf{A} which induce the given abstract kernel Ψ . We first fix some notation and develop necessary preliminaries for this purpose.

Remark 2.9. Suppose \mathbf{A} is a covariant E -diagram in an arbitrary category C which factors through $\mathbf{D}(B(E))$. That is, there is a (necessarily unique) functor $\hat{\mathbf{A}} : \mathbf{D}(B(E)) \rightarrow C$ such that the diagram

$$\begin{array}{ccc}
 \underline{C}(E) & \xrightarrow{\mathbf{A}} & C \\
 \searrow^{\pi_1} & & \swarrow_{\hat{\mathbf{A}}} \\
 & \mathbf{D}(B(E)) &
 \end{array}
 \tag{2.22}$$

is commutative, where $\pi_1 : (e, c(e_0, e_1, \dots, e_n)) \rightarrow [e, e_0e_1 \cdots e_n, e_n e_{n-1} \cdots e_0] : \underline{C}(E) \rightarrow \mathbf{D}(B(E))$ is the composite $\underline{C}(E) \xrightarrow{\bar{\varepsilon}_{B(E)}} C(B(E)) \rightarrow \mathbf{D}(B(E))$. In this case, for any idempotent-separating homomorphism $\mu : S \rightarrow \text{Reg}_E(\mathbf{A})$ with $\mu\theta_S = \theta_A$, the associated covariant functor $\mathbf{A}_\mu : C(S) \rightarrow C : (\mathbf{A}_\mu)_e = \mathbf{A}_e; e \in C(S)$ and $\mathbf{A}_\mu(e, x, x') = (\phi_{xx'}(x, x'))^{-1} \mathbf{A}(e, xx')$ for each morphism $(e, x, x') : e \rightarrow f$ of $C(S)$ factors through $\mathbf{D}(S)$. We denote the functor

$e \rightarrow \mathbf{A}_e; [e, x, x'] \rightarrow \mathbf{A}(e, xx')\phi_{xx'}(x, x') : \mathbf{D}(S) \rightarrow C$ by $\overline{\mathbf{A}}_\mu$ itself so that the diagram

$$\begin{array}{ccc}
 \underline{C}(E) & & \\
 \varepsilon_S \downarrow & \searrow \mathbf{A} & \\
 \mathbf{C}(S) & & C \\
 \downarrow & \nearrow \mathbf{A}_\mu & \\
 \mathbf{D}(S) & &
 \end{array} \tag{2.23}$$

is commutative.

If $e, f \in E$, then for any $h \in S(e, f)$, $(e, c(eh, h, hf))$ is a morphism from e to hf in $\underline{C}(E)$.

We write

$$D(h, e, f) = (e, c(eh, h, hf)). \tag{2.24}$$

If $f\omega^l e$, then we write

$$L(e, f) = (e, c(ef, f)). \tag{2.25}$$

Note that $L(e, f) = D(f, e, f)$. Also note that

$$\pi_1 D(h, e, f) = [e, ef, h] : e \rightarrow hf, \quad \pi_1 L(e, f) = [e, ef, f] : e \rightarrow f, \tag{2.26}$$

where $\pi_1 : \underline{C}(E) \rightarrow \mathbf{D}(B(E))$ is as in Remark 2.9.

LEMMA 2.10. *Let $\mathbf{A} : \underline{C}(E) \rightarrow \mathbf{GR}$ be an E -diagram that factors through $\mathbf{D}(B(E))$. Let e, f, g, \dots denote arbitrary elements of E .*

(i) If $g\omega^l f\omega^l e$, then

$$\mathbf{A}(L(e, f)L(f, g)) = \mathbf{A}(L(e, g)). \tag{2.27}$$

(ii) If $f\omega^l e, h \in S(f, g)$, then $h \in S(e, hg)$ and

$$\mathbf{A}(L(e, f)D(h, f, g)) = \mathbf{A}(D(h, e, hg)). \tag{2.28}$$

If, in addition, $h \in S(e, g)$ (this happens, e.g., fLe), then

$$\mathbf{A}(L(e, f)D(h, f, g)) = \mathbf{A}(D(h, e, g)). \tag{2.29}$$

(iii) If $h \in S(e, f)$ and $g \in S(e, k)$, with $k\omega f$, then $gf \in S(hf, k)$ and

$$\mathbf{A}(D(h, e, f)D(gf, hf, k)) = \mathbf{A}(D(g, e, k)). \tag{2.30}$$

(iv) If $h \in S(e, f), g\omega^l e, g\omega^r f$, then $g \in S(e, gf), gf\omega^l hf$, and

$$\mathbf{A}(D(h, e, f)L(hf, gf)) = \mathbf{A}(D(g, e, gf)). \tag{2.31}$$

If, in addition, $g \in S(e, f)$, then

$$\mathbf{A}(D(h, e, f)L(hf, gf)) = \mathbf{A}(D(g, e, f)). \tag{2.32}$$

(v) If $h \in S(e, f)$, $g\mathbf{R}f$, then $h \in S(e, g)$, $hf \in S(hf, g)$, and

$$\mathbf{A}(D(h, e, f)D(hf, hf, g)) = \mathbf{A}(D(h, e, g)). \tag{2.33}$$

(vi) If $f\omega^l e, g \in S(e, n)$, $h \in S(f, eg)$, $m \in S(f, n)$, $k \in S(h(eg), n)$, with $mn\mathbf{L}kn$, then

$$\mathbf{A}(D(h, f, eg)D(k, h(eg), n)) = \mathbf{A}(D(m, f, n)L(mn, kn)). \tag{2.34}$$

Proof. By Remark 2.9, it is sufficient to prove (i)-(ii) replacing \mathbf{A} by the functor $\pi_1 : \underline{\mathbf{C}}(E) \rightarrow \mathbf{D}(B(E))$. We frequently use (1.2) to prove the lemma.

(i) Using (1.2) we get $\pi_1(L(e, f)L(f, g)) = \pi_1(L(e, f)\pi_1 L(f, g)) = [e, ef, f][f, fg, g] = [e, (ef)(fg), g] = [e, eg, g] = \pi_1(L(e, g))$. This proves (i).

(ii) Let $f\omega^l e$ and $h \in S(f, g)$. Then clearly $h \in S(e, hg)$, and $\pi_1(L(e, f)D(h, f, g)) = [e, ef, f][f, fg, h] = [e, (ef)(fg), h] = [e, e(hg), h] = \pi_1(D(h, e, hg))$. If $h \in S(e, g)$, then $D(h, e, g) = D(h, e, hg)$. Therefore, the second statement follows from the first.

(iii) Clearly $gf \in S(hf, k)$. Now $\pi_1(D(h, e, f)D(gf, hf, k)) = [e, ef, h][hf, (hf)k, gf] = [e, (ef)((hf)k), (gf)h] = [e, ek, gh] = [e, ek, g] = \pi_1(D(g, e, k))$, since $(ef)((hf)k) = ek$, and $(gf)h = gh$.

(iv) Clearly $g \in S(e, gf)$ and $gf\omega^l hf$. By taking $k = gf$ in (iii) and observing $D(gf, hf, gf) = L(hf, gf)$, we get $\pi_1(D(h, e, f)L(hf, gf)) = \pi_1(D(g, e, gf))$. The last relation follows from this since $D(g, e, gf) = D(g, e, f)$, if $g \in S(e, f)$.

(v) Let $h \in S(e, f)$, $g\mathbf{R}f$. Then, by [13, Proposition 2.12], $S(e, f) = S(e, g)$ and so $h \in S(e, g)$. Clearly $hf \in S(hf, g)$. Further, since $(hf)h = h(fh) = h$, we get $\pi_1(D(h, e, f)D(hf, hf, g)) = [e, ef, h][hf, (hf)g, hf] = [e, (ef)((hf)g), h] = [e, eg, h] = \pi_1(D(h, e, g))$.

(vi)

$$\begin{aligned} &\pi_1(D(h, f, eg)D(k, h(eg), n)) \\ &= [f, f(eg), h][h(eg), (h(eg)n, k)] \\ &= [f, (f(eg))((h(eg))n), kh] \\ &= [f, (f(eg))n, kh] \quad \text{since } (eg)h = h, (fh)(eg) = f(eg) \\ &= [f, fn, kh] \quad \text{since } (eg)n = en, fe = f \\ &= [f, f(nm)n, kh] \quad \text{since } fmn = fn, nm = m \\ &= [f, (fn)(mn), kh] \\ &= [f, (fn)(mn), (kn)m] \\ &= [f, fn, m][mn, mn, kn] \\ &= \pi_1(D(m, f, n)L(mn, kn)). \end{aligned} \tag{2.35}$$

The proof of the lemma is complete. □

We fix once and for all a map $*$: $S \rightarrow S$ such that

- (i) $x^* \in V(x)$ for every $x \in S$,
- (ii)

$$x^* \in \mathbf{H}_e \quad \text{if } x \in \mathbf{H}_e. \tag{2.36}$$

Suppose $\Psi : S \rightarrow \text{Reg}_E(\mathbf{A})/\text{Inn}_E(\mathbf{A})$ is an abstract kernel, and let $\sigma : S \rightarrow \text{Reg}_E(\mathbf{A})$ be a map such that $(x)\sigma \in (x)\Psi$, for every $x \in S$. By Remark 2.9(ii) and by [13], each $(x)\sigma$ has a unique representative in $\mathbf{G}(\mathbf{A})$ with domain xx^* and range x^*x . We denote this morphism by $(\alpha(x), \phi(x)) : xx^* \rightarrow x^*x$ so that $[\alpha(x), \phi(x)] = (x)\sigma$; recall by Remark 2.9(ii), $(h)\alpha(x) = x^*hx$ for all $h \in \omega(xx^*)$. Using σ we will define a biaction of S on the disjoint union

$$\mathbf{A} = \bigcup_{x \in S} \mathbf{A}_x, \quad \text{where } \mathbf{A}_x = \mathbf{A}_{x^*x}. \tag{2.37}$$

For $x, y \in S$, define

$$a \rightarrow x \bullet a : \mathbf{A}_y \rightarrow \mathbf{A}_{xy}, \quad a \rightarrow a \bullet x : \mathbf{A}_y \rightarrow \mathbf{A}_{yx} \tag{2.38}$$

by

$$x \bullet a = a\mathbf{A}(L(y^*y, (xy)^*xy)), \tag{2.39}$$

$$a \bullet x = a\mathbf{A}(D(h, y^*y, xx^*))\phi_{hxx^*}(x)\mathbf{A}(L(x^*hx, (yx)^*yx)), \tag{2.40}$$

where $h \in S(y^*y, xx^*)$ and $\phi_{hxx^*}(x) : \mathbf{A}_{hxx^*} \rightarrow \mathbf{A}_{(hxx^*)\alpha(x)=x^*hx}$ is the component of $\phi(x)$ at $hxx^* \in \omega(xx^*)$. If $k \in S(y^*y, xx^*)$ is any other element, then the following diagram commutes:

$$\begin{array}{ccccc}
 & & \mathbf{A}_{hxx^*} & \xrightarrow{\phi_{hxx^*}(x)} & \mathbf{A}_{x^*hx} & & \\
 & \nearrow \mathbf{A}(D(h, y^*y, xx^*)) & \downarrow \mathbf{A}(L(hxx^*, kxx^*)) & & \downarrow \mathbf{A}(L(x^*hx, (yx)^*yx)) & \searrow & \\
 \mathbf{A}_y = \mathbf{A}_{y^*y} & & & & & & \mathbf{A}_{(yx)^*yx} = \mathbf{A}_{yx} \\
 & \searrow \mathbf{A}(D(k, y^*y, xx^*)) & \downarrow \phi_{kxx^*}(x) & & \downarrow \mathbf{A}(L(x^*kx, (yx)^*yx)) & \nearrow & \\
 & & \mathbf{A}_{kxx^*} & \xrightarrow{\phi_{kxx^*}(x)} & \mathbf{A}_{x^*kx} & &
 \end{array} \tag{2.41}$$

The first triangle is commutative by Lemma 2.10(iv) and the last triangle is commutative by Lemma 2.10(i), since $(yx)^*yx\omega^l x^*kx\omega^l x^*hx$. Finally the commutativity of the rectangle follows from the naturality of $\phi(x)$. Hence, $a \bullet x$ does not depend on the choice of h . Clearly $a \rightarrow a \bullet x$ and $a \rightarrow x \bullet a$ are homomorphisms of groups.

The following lemma explains to what extent the biaction of S on \mathbf{A} depends on σ .

LEMMA 2.11. Suppose $\sigma, \sigma' : S \rightarrow \text{Reg}_E(\mathbf{A})$ are maps, with $(x)\sigma, (x)\sigma' \in (x)\Psi$, and let \bullet and \spadesuit be the corresponding biactions of S on \mathbf{A} . Then

(i)

$$x\spadesuit a = x \bullet a \quad x \in S, a \in \mathbf{A}_y, \tag{2.42}$$

(ii) there exists a map $\beta : S \rightarrow \mathbf{A}$, with $(x)\beta \in \mathbf{A}_x$, such that

$$\begin{aligned} (x)\sigma' &= (x)\sigma((x)\beta)\eta, \\ a\spadesuit x &= (a \bullet x)(y \bullet (x)\beta)\eta, \quad x \in S, a \in \mathbf{A}_y, \end{aligned} \tag{2.43}$$

where $\eta : \mathbf{A}_{yx} (= \mathbf{A}_{(yx)^*yx}) \rightarrow \text{Reg}_E(\mathbf{A})$ is as in (2.6).

Proof. (i) is clear, since $x\spadesuit a = a\mathbf{A}(L(y^*y, (xy)^*xy)) = x \bullet a$.

(ii) Since $(x)\sigma, (x)\sigma'$ belong to the same class $(x)\Psi$, by Proposition 2.5, there must be elements $(x)\beta \in \mathbf{A}_x$ such that $(x)\sigma' = (x)\sigma((x)\beta)\eta$. Let $(\alpha(x), \phi(x)) : xx^* \rightarrow x^*x$ and $(\overline{\alpha(x)}, \overline{\phi(x)}) : xx^* \rightarrow x^*x$ be unique representatives of $(x)\sigma$ and $(x)\sigma'$ with domain xx^* and range x^*x .

$$\begin{aligned} \implies [\overline{\alpha(x)}, \overline{\phi(x)}] &= [\alpha(x), \phi(x)][1_{x^*x}, \eta^{(x)\beta}] \\ \implies \overline{\phi_e}(x) &= \{\phi(x)(\underline{C}(\alpha(x))\eta^{(x)\beta})\}_e = \phi_e(x)\eta^{(x)\beta}_{x^*ex}, \end{aligned} \tag{2.44}$$

for every $e \in \omega(xx^*)$. Hence, for $h \in S(y^*y, xx^*)$,

$$\begin{aligned} a\spadesuit x &= a\mathbf{A}(D(h, y^*y, xx^*))\overline{\phi_{hxx^*}}(x)\mathbf{A}(L(x^*hx, (yx)^*yx)) \\ &= \{\{a\mathbf{A}(D(h, y^*y, xx^*))\phi_{hxx^*}(x)\}\eta^{(x)\beta}_{x^*hx}\}\mathbf{A}(L(x^*hx, (yx)^*yx)) \\ &= [(x)\beta\mathbf{A}(L(x^*x, x^*hx))\mathbf{A}(L(x^*hx, (yx)^*yx))]^{-1} \\ &\quad \times [a\mathbf{A}(D(h, y^*y, xx^*))\phi_{hxx^*}(x)\mathbf{A}(L(x^*hx, (yx)^*yx))] \\ &\quad \times [(x)\beta\mathbf{A}(L(x^*x, x^*hx))\mathbf{A}(L(x^*hx, (yx)^*yx))] \\ &= [(x)\beta\mathbf{A}(L(x^*x, (yx)^*yx))]^{-1}(a \bullet x)[(x)\beta\mathbf{A}(L(x^*x, (yx)^*yx))] \\ &\quad \text{by Lemma 2.10(i), since } (yx)^*yx\omega^l x^*hx\omega^l x^*x \\ &= (y \bullet (x)\beta)^{-1}(a \bullet x)(y \bullet (x)\beta) \\ &= (a \bullet x)(y \bullet (x)\beta)\eta. \end{aligned} \tag{2.45}$$

□

Definition 2.12. Let $\Psi : S \rightarrow \text{Reg}_E(\mathbf{A})/\text{Inn}_E(\mathbf{A})$ be an abstract kernel. Let $\sigma : S \rightarrow \text{Reg}_E(\mathbf{A})$ and $p : S \times S \rightarrow \mathbf{A}$, $(x, y)p \in \mathbf{A}_{xy}$, be maps such that

(i)

$$(x)\sigma \in (x)\Psi, \tag{2.46}$$

(ii)

$$(x)\sigma(y)\sigma = (xy)\sigma((x,y)p)\eta, \tag{2.47}$$

(iii)

$$(xy,z)p((x,y)p \bullet z) = (x,yz)p(x \bullet (y,z))p, \tag{2.48}$$

where $\eta : \mathbf{A}_{xy} [= \mathbf{A}_{(xy)^*xy}] \rightarrow \text{Reg}_E(\mathbf{A})$ is as in (2.6) and the biaction \bullet of S on \mathbf{A} is with respect to the map σ . Then the pair (σ, p) is called a *crossed pair*.

Let Ψ be an abstract kernel and let σ, p be maps satisfying Definition 2.12(i) and (ii). In the next two lemmas, we establish some of the essential properties of the biaction \bullet of S on \mathbf{A} induced by σ .

LEMMA 2.13. *Let $\Psi : S \rightarrow \text{Reg}_E(\mathbf{A})/\text{Inn}_E(\mathbf{A})$ be an abstract kernel. Let (σ, p) satisfy Definition 2.12(i) and (ii) and let \bullet denote the biaction of S on \mathbf{A} induced by σ .*

(i) If $(\alpha(e), \phi(e)) : e \rightarrow e$ is a representative of $(e)\sigma$ with domain and range e , then $\phi_e(e)$ coincides with the inner automorphism defined by $(e,e)p$. More generally, if $e_1\omega^r e$, then for any $a \in \mathbf{A}_e = \mathbf{A}_{e_1}, a \bullet e_1 = (e_1, e_1)p^{-1}a\mathbf{A}(e, c(e_1e, e_1))(e_1, e_1)p$.

(ii) If $x \in S, e \in E(S)$, with $ex = x$, then for $a \in \mathbf{A}_x, e \bullet a = a$. If (σ, p) also satisfies (2.48), then for $y \in S, e \in E(S), (e,e)p \bullet y = (e,y)p^{-1}(e,ey)p(e,y)p$.

Proof. (i) For $e \in E(S), (e)\sigma(e)\sigma = (e)\sigma((e,e)p)\eta \Rightarrow (e)\sigma = ((e,e)p)\eta \Rightarrow \phi_e(e) = \eta_e^{(e,e)p}$, the inner automorphism defined by the element $(e,e)p$.

If $e_1\omega^r e$, then $e_1e \in S(e, e_1)$ and for $a \in \mathbf{A}_e$, by (2.40),

$$\begin{aligned} a \bullet e_1 &= a\mathbf{A}(D(e_1e, e, e_1))\phi_{e_1}(e_1) = a\mathbf{A}(e, c(e_1e, e_1))\eta_{e_1}^{(e_1, e_1)p} \\ &= (e_1, e_1)p^{-1}a\mathbf{A}(e, c(e_1e, e_1))(e_1, e_1)p. \end{aligned} \tag{2.49}$$

(ii) Clearly $e \bullet a = a\mathbf{A}(L(x^*x, (ex)^*ex)) = a\mathbf{A}(L(x^*x, x^*x)) = a$. To prove the last assertion, let $y \in S, e \in E(S)$. Then, by (2.48), $(e,y)p((e,e)p \bullet y) = (e,ey)p(e \bullet (e,y))p$ or $(e,e)p \bullet y = (e,y)p^{-1}(e,ey)p(e,y)p$, since $e(ey) = ey$ implies $(e \bullet (e,y))p = (e,y)p$. \square

LEMMA 2.14. *Let Ψ, σ, p be as in the first paragraph of Lemma 2.13. Then*

(i)

$$x \bullet (y \bullet a) = xy \bullet a, \quad x, y \in S, a \in \mathbf{A}_z, \tag{2.50}$$

(ii)

$$x \bullet (b \bullet z) = (x \bullet b) \bullet z, \quad x, z \in S, b \in \mathbf{A}_y, \tag{2.51}$$

(iii)

$$\begin{aligned} (d \bullet y) \bullet z &= (d \bullet yz)(x \bullet (y,z)p)\eta \\ &= (x \bullet (y,z)p)^{-1}(d \bullet yz)(x \bullet (y,z)p) \quad y, z \in S, d \in \mathbf{A}_x. \end{aligned} \tag{2.52}$$

Proof. (i) is immediate from Lemma 2.10(i), since $(xyz)^*xyz\omega^l(yz)^*yz\omega^lz^*$.

As before for $x \in S$, let $(\alpha(x), \phi(x)) : xx^* \rightarrow x^*x$ denote the unique representative of $(x)\sigma$ with domain xx^* and range x^*x . Let $x, y, z \in S$. Choose $h \in S(x^*x, yy^*)$, $h_2 \in S(y^*y, zz^*)$. Then $(xy)^*xyLy^*hy$ and $yz(yz)^*Ryh_2y^*$. Let $h_1 \in S((xy)^*xy, zz^*) = S(y^*hy, zz^*)$, $h_3 \in S(x^*x, yz(yz)^*) = S(x^*x, yh_2y^*)$. Since $\phi(z) : \mathbf{A}^{zz^*} \rightarrow \mathbf{A}^{z^*z}\underline{C}(\alpha(z))$ is a natural isomorphism, the diagram

$$\begin{CD}
 \mathbf{A}_{h_2zz^*} @>\phi_{h_2zz^*}(z)>> \mathbf{A}_{(h_2zz^*)\alpha(z)=z^*h_2z} \\
 @V\mathbf{A}(L(h_2zz^*, h_1zz^*))VV @VV\mathbf{A}(L(z^*h_2z, z^*h_1z))V \\
 \mathbf{A}_{h_1zz^*} @>\phi_{h_1zz^*}(z)>> \mathbf{A}_{(h_1zz^*)\alpha(z)=z^*h_1z}
 \end{CD} \tag{2.53}$$

is commutative. Now, using Lemma 2.10(i) twice, we get

$$\begin{aligned}
 x \bullet (b \bullet z) &= b\mathbf{A}(D(h_2, y^*y, zz^*))\phi_{h_2zz^*}(z)\mathbf{A}(L(z^*h_2z, (yz)^*yz)) \\
 &\quad \times \mathbf{A}(L((yz)^*yz, (xyz)^*xyz)) \\
 &= b\mathbf{A}(D(h_2, y^*y, zz^*))\phi_{h_2zz^*}(z)\mathbf{A}(L(z^*h_2z, z^*h_1z))\mathbf{A}(L(z^*h_1z, (xyz)^*xyz)) \\
 &= b\mathbf{A}[(D(h_2, y^*y, zz^*)L(h_2zz^*, h_1zz^*))]\phi_{h_1zz^*}(z)\mathbf{A}(L(z^*h_1z, (xyz)^*xyz)) \\
 &= b\mathbf{A}(D(h_1, y^*y, h_1zz^*))\phi_{h_1zz^*}(z)\mathbf{A}(L(z^*h_1z, (xyz)^*xyz)) \\
 &\quad \text{by Lemma 2.10(iv), since } h_2 \in S(y^*y, zz^*), h_1\omega^l y^*y, h_1\omega^r zz^* \\
 &= b\mathbf{A}(L(y^*y, (xy)^*xy))\mathbf{A}(D(h_1, (xy)^*xy, zz^*))\phi_{h_1zz^*}(z) \\
 &\quad \times \mathbf{A}(L(z^*h_1z, (xyz)^*xyz)) \\
 &\quad \text{by Lemma 2.10(ii), since } (xy)^*xy\omega^l y^*y \text{ and } h_1 \in S((xy)^*xy, zz^*) \\
 &= (x \bullet b) \bullet z.
 \end{aligned} \tag{2.54}$$

Hence, the proof of (ii) is complete. To prove (iii), consider the diagram

$$\tag{2.55}$$

where

$$\begin{aligned}
 C_1 &= \mathbf{A}(L(y^*hy, (xy)^*xy)), & C_2 &= \mathbf{A}(D(h_1, (xy)^*xy, zz^*)), \\
 C_3 &= \mathbf{A}(D(h, x^*x, yy^*)), & C_4 &= \phi_{hyy^*}(y), & C_5 &= \mathbf{A}(D(h_1, y^*hy, zz^*)), \\
 C_6 &= \phi_{h_1zz^*}(h), & C_7 &= \mathbf{A}(D(h_3yy^*, hyy^*, yh_2y^*)), \\
 C_8 &= \mathbf{A}(D(y^*h_3y, y^*hy, y^*yh_2)), & C_9 &= \mathbf{A}(L(h_1zz^*, h_2y^*h_3yh_2zz^*)), \\
 C_{10} &= \mathbf{A}(L(z^*h_1z, z^*h_2y^*h_3yz)), & C_{11} &= \mathbf{A}(L(z^*h_1z, (xyz)^*xyz)), \\
 C_{12} &= \mathbf{A}(D(h_3, x^*x, yh_2y^*)), & C_{13} &= \phi_{h_3yh_2y^*}(y), \\
 C_{14} &= \mathbf{A}(D(h_2y^*h_3yh_2, y^*h_3yh_2, zz^*)), & C_{15} &= \phi_{h_2y^*h_3yh_2zz^*}(z), \\
 C_{16} &= \mathbf{A}(D(h_3, x^*x, yz(yz)^*)), & C_{17} &= \mathbf{A}(D(h_3yz(yz)^*, h_3yz(yz)^*, yh_2y^*)), \\
 C_{18} &= \mathbf{A}(L(z^*h_3y^*h_3yz, (yz)^*h_3yz)), & C_{19} &= \mathbf{A}(L((yz)^*h_3yz, (xyz)^*xyz)), \\
 C_{20} &= \phi_{h_3yz(yz)^*}(yz)\eta_{(yz)h_3yz}^{(y,z)P\mathbf{A}((yz)^*yz, (yz)^*h_3yz)}.
 \end{aligned} \tag{2.56}$$

The commutativity of the diagram I follows from Lemma 2.10(ii), since $(xy)^*xy\mathbf{L}y^*hy$ and $h_1 \in S((xy)^*xy, zz^*) = S(y^*hy, zz^*)$. Since $yh_2y^*\omega yy^*$, by Lemma 2.10(iii), $h_3yy^* \in S(hyy^*, yh_2y^*)$ and the diagram II is commutative. The diagrams III and V are commutative, since $\phi(y)$ and $\phi(z)$ are natural isomorphisms. Next we show that the diagram IV is commutative. Now

$$y^*h_3yh_2\omega^l h_2 \implies y^*h_3yh_2\mathbf{L}h_2y^*h_3yh_2 \implies h_2y^*h_3yh_2 \in S(y^*h_3yh_2, zz^*). \tag{2.57}$$

Also

$$h_3\omega^r yh_2y^* \implies h_1zz^*h_2y^*h_3 = h_1(xy)^*xh_3 \implies h_1zz^*\mathbf{L}h_2y^*h_3yzz^*, \tag{2.58}$$

since $zz^*h_2 = h_2$ and $h_3yh_1zz^* = h_3x^*xyh_1zz^* = h_3yzz^*$. Take $e = y^*y$, $f = y^*hy$, $g = h_2$, $h = y^*h_3y$, $k = h_2y^*h_3yh_2$, $m = h_1$, $n = zz^*$. The commutativity of the diagram IV now follows from Lemma 2.10(vi). Since $yz(yz)^*\mathbf{R}yh_2y^*$, the commutativity of the diagram VI follows from Lemma 2.10(v). Since $z^*h_1z\mathbf{L}z^*h_2y^*h_3yz\mathbf{L}(yz)^*h_3yz\mathbf{L}(xyz)^*xyz$, the diagram VIII is commutative by Lemma 2.10(i). Finally we establish the commutativity of the diagram VII. Put $c_1 = c(y^*yh_2, h_2, h_2zz^*)$, $c_2 = c(yz(yz)^*, yh_2y^*)$, $c_3 = c(z^*h_2z, (yz)^*yz)$, $(\alpha_1, \phi_1) = yh_2y^* * (\alpha(y), \phi(y))$, $(\alpha_2, \phi_2) = (\alpha(z), \phi(z)) * z^*h_2z$. Then

$$\begin{aligned}
 & [(\alpha(yz), \phi(yz))(1_{(yz)^*yz}, \eta^{(y,z)P})] \\
 &= [\alpha(yz), \phi(yz)][1_{(yz)^*yz}, \eta^{(y,z)P}] \\
 &= (yz)\sigma((y, z)P)\eta \\
 &= (y)\sigma(z)\sigma \\
 &= [\alpha(y), \phi(y)][\alpha(z), \phi(z)] \\
 &= [((\alpha(y), \phi(y)) * y^*yh_2)\varepsilon(c_1)(h_2zz^* * (\alpha(z), \phi(z)))] \text{ by [10]} \\
 &= [(\alpha_1, \phi_1)\varepsilon(c_1)(\alpha_2, \phi_2)] \text{ by [13, Proposition 3.2].}
 \end{aligned} \tag{2.59}$$

Since $yz(yz)^* \mathbf{R}y h_2 y^*$ and $z^* h_2 z \mathbf{L}(yz)^* yz$, (1.9) implies that

$$\begin{aligned} & (\alpha(yz), \phi(yz)) (1_{(yz)^* yz}, \eta^{(y,z)p}) \\ &= \varepsilon(c_2) (\alpha_1, \phi_1) \varepsilon(c_1) (\alpha_2, \phi_2) \varepsilon(c_3) \\ &= (\alpha^{c_2}, \phi^{c_2}) (\alpha_1, \phi_1) (\alpha^{c_1}, \phi^{c_1}) (\alpha_2, \phi_2) (\alpha^{c_3}, \phi^{c_3}). \end{aligned} \tag{2.60}$$

Therefore, the component at e of the natural isomorphism defined by the left-hand side coincides with the component at e of the natural isomorphism defined by the right-hand side for each $e \in \omega(yz(yz)^*)$. In particular, by taking $e = h_3 yz(yz)^*$ and noting that $\phi_e^{c_2} = \mathbf{A}(D(e, e, y h_2 y^*))$ and $\phi_e^{c_1} = \mathbf{A}(D(h_2 y^* h_3 y h_2, y^* h_3 y h_2, z z^*))$, we obtain the commutativity of the diagram VII. As the interior diagrams are commutative, the outer diagram is commutative. Hence, for $d \in \mathbf{A}_x = \mathbf{A}_{x^* x}$,

$$\begin{aligned} (d \bullet y) \bullet z &= d \mathbf{A}(D(h, x^* x, y y^*)) \phi_{h y y^*}(y) \mathbf{A}(L(y^* h y, (x y)^* x y)) \\ &\quad \times \mathbf{A}(D(h_1, (x y)^* x y, z z^*)) \phi_{h_1 z z^*}(z) \mathbf{A}(L(z^* h_1 z, (x y z)^* x y z)) \\ &= d \mathbf{A}(D(h_3, x^* x, y z(yz)^*)) \phi_e(yz) \mathbf{A}(L((yz)^* h_3 yz, (x y z)^* x y z)) \eta_{(x y z)^* x y z}^{(x \bullet (y,z)p)} \\ &= (x \bullet (y,z)p)^{-1} (d \bullet yz) (x \bullet (y,z)p). \end{aligned} \tag{2.61}$$

With these preliminaries we are now in a position to describe the extensions of S by \mathbf{A} which induce the given abstract kernel Ψ . □

THEOREM 2.15. *Let $\Psi : S \rightarrow \text{Reg}_E(\mathbf{A})/\text{Inn}_E(\mathbf{A})$ be an abstract kernel and let (σ, p) be a crossed pair. Let*

$$T_p = \{(x, a) : x \in S, a \in \mathbf{A}_x\}. \tag{2.62}$$

Define a multiplication on T_p by

$$(x, a)(y, b) = (xy, (x, y)p(a \bullet y)(x \bullet b)). \tag{2.63}$$

Then T_p is a regular semigroup with

$$E(T_p) = \{(e, (e, e)p^{-1}) : e \in E(S)\}. \tag{2.64}$$

The map $\pi_p : T_p \rightarrow S$ defined by $(x, a)\pi_p = x$, is an idempotent-separating homomorphism of T_p onto S . For each $e \in E = E(S)$, define $(U_p)_e : \mathbf{A}_e \rightarrow \mathbf{A}_e^{\pi_p}$ by

$$(a)(U_p)_e = (e, (e, e)p^{-1}a). \tag{2.65}$$

Then $U_p : e \rightarrow (U_p)_e$ defines a natural isomorphism between \mathbf{A} and \mathbf{A}^{π_p} . The triple (T_p, π_p, U_p) is an extension of S by \mathbf{A} .

Proof. For $(x, a), (y, b), (z, c) \in T_p$, by (2.48), Lemma 2.14, we easily prove that $((x, a)(y, b))(z, c) = (x, a)((y, b)(z, c))$. So the multiplication is associative. For each $e \in E(S)$, $(e, (e, e)p^{-1})(e, (e, e)p^{-1}) = (e, (e, e)p((e, e)p^{-1} \bullet e)(e \bullet (e, e)p^{-1})) = (e, (e, e)p^{-1})$, since

$(e, e)p^{-1} \bullet e = e \bullet (e, e)p^{-1} = (e, e)p^{-1}$, by Lemma 2.13(ii). Hence, $(e, (e, e)p^{-1}) \in E(T_p)$. On the other hand, $(e, a)(e, a) = (e, a) \Rightarrow (ee, (e, e)p(a \bullet e)(e \bullet a)) = (e, a) \Rightarrow ee = e$ and $(e, e)p(a \bullet e)(e \bullet a) = a \Rightarrow e \in E(S)$ and $(e, e)p = a^{-1} \bullet e = (e, e)p^{-1}a^{-1}(e, e)p$ (by Lemma 2.13(i) and (ii)) $\Rightarrow e \in E(S)$ and $a = (e, e)p^{-1}$.

Hence, $E(T_p) = \{(e, (e, e)p^{-1}) : e \in E(S)\}$. To prove T_p is a regular semigroup, take any $(x, a) \in T_p$ and let y be an inverse of x in S . Put $b = y \bullet ((xy, xy)p(x, y)p(a \bullet y))^{-1}$. Then $(y, b) \in T_p$, and $x \bullet b = xy \bullet ((xy, xy)p(x, y)p(a \bullet y))^{-1}$ (by Lemma 2.14(i)) $= ((xy, xy)p(x, y)p(a \bullet y))^{-1}$ (by Lemma 2.13(ii)). Then

$$\begin{aligned} (x, a)(y, b) &= (xy, (x, y)p(a \bullet y)(x \bullet b)) \\ &= (xy, (x, y)p(a \bullet y)(a \bullet y)^{-1}(x, y)p^{-1}(xy, xy)p^{-1}) \\ &= (xy, (xy, xy)p^{-1}). \end{aligned} \tag{2.66}$$

Therefore, $(x, a)(y, b)(x, a) = (xy, (xy, xy)p^{-1})(x, a) = (xyx, (xy, x)p((xy, xy)p^{-1} \bullet x)(xy \bullet a)) = (x, a)$, since by Lemma 2.13(ii), $xy \bullet a = a$ and $(xy, xy)p \bullet x = (xy, x)p$ and $(y, b)(x, a)(y, b) = (y, b)(xy, (xy, xy)p^{-1}) = (y, (y, xy)p(b \bullet xy)(y \bullet (xy, xy)p^{-1})) = (y, b)$, since

$$\begin{aligned} b \bullet xy &= [y \bullet ((a^{-1} \bullet y)(x, y)p^{-1}(xy, xy)p^{-1})] \bullet xy \\ &= y \bullet [((a^{-1} \bullet y)(x, y)p^{-1}(xy, xy)p^{-1}) \bullet xy] \quad \text{by Lemma 2.14(ii)} \\ &= y \bullet [((a^{-1} \bullet y) \bullet xy)((x, y)p^{-1} \bullet xy)((xy, xy)p^{-1} \bullet xy)] \\ &= y \bullet ((x \bullet (y, xy)p^{-1})(a^{-1} \bullet y)(x \bullet (y, xy)p)((x, y)p^{-1} \bullet xy)(xy, xy)p^{-1}) \\ &\quad \text{by Lemmas 2.14(iii) and 2.13(ii)} \\ &= y \bullet ((x \bullet (y, xy)p^{-1})(a^{-1} \bullet y)(x, y)p^{-1}) \quad \text{using (2.48) for the triple } x, y, xy \\ &= (yx \bullet (y, xy)p^{-1})(y \bullet (a^{-1} \bullet y)(x, y)p^{-1}) \quad \text{by Lemma 2.14(i)} \\ &= (y, xy)p^{-1}(y \bullet (a^{-1} \bullet y)(x, y)p^{-1}) \quad \text{by Lemma 2.13(ii)}. \end{aligned} \tag{2.67}$$

Hence, (y, b) is an inverse of (x, a) , and T_p is a regular semigroup. The map $\pi_p : T_p \rightarrow S$, $(x, a)\pi_p = x$, is clearly an idempotent-separating homomorphism from T_p onto S with $\mathbf{A}_e^{\pi p} = \{(e, a) : a \in \mathbf{A}_e\}$ for each $e \in E$. The map $U_e = (U_p)_e : \mathbf{A}_e \rightarrow \mathbf{A}_e^{\pi p}$ defined by (2.65) is clearly a bijection. By Lemma 2.13, it is clear that U_e is also a homomorphism.

We next show that the isomorphisms U_e define a natural isomorphism $U_p : \mathbf{A} \rightarrow \mathbf{A}^{\pi p}$. We must show that for each morphism $(e, c(e_0, \dots, e_n)) : e \rightarrow f$ in $\underline{C}(E)$, the diagram

$$\begin{array}{ccc} \mathbf{A}_e & \xrightarrow{U_e} & \mathbf{A}_e^{\pi p} \\ \mathbf{A}(e, c) \downarrow & & \downarrow \mathbf{A}^{\pi p}(e, c) \\ \mathbf{A}_f & \xrightarrow{U_f} & \mathbf{A}_f^{\pi p} \end{array} \tag{2.68}$$

is commutative. Since $(e, c(e_0, \dots, e_n)) = (e, e_0)(e_0, c(e_0, e_1)) \cdots (e_{n-1}, c(e_{n-1}, e_n))$, it is enough to prove the commutativity of the diagram for morphisms of the form $(e, c(e, f))$, with $e \geq f$ or $e(\mathbf{R} \cup \mathbf{L})f$.

Case 1 ($e \geq f$). Let $a \in \mathbf{A}_e$. Then, since

$$\begin{aligned} \mathbf{A}^{\pi p}(e, f) &= \text{Ker } \pi_p[(e, (e, e)p^{-1}), (f, (f, f)p^{-1}), (f, (f, f)p^{-1})], \\ (a)U_e\mathbf{A}^{\pi p}(e, f) &= (e, (e, e)p^{-1}a)\mathbf{A}^{\pi p}(e, f) \\ &= (f, (f, f)p^{-1})(e, (e, e)p^{-1}a)(f, (f, f)p^{-1}) \\ &= (e, (e, e)p^{-1}a)(f, (f, f)p^{-1}) \quad \text{by Definition 1.1} \\ &= (f, (e, f)p((e, e)p^{-1}a \bullet f)(e \bullet (f, f)p^{-1})) \\ &= (f, (e, f)p((e, e)p^{-1} \bullet f)(a \bullet f)(f, f)p^{-1}) \\ &\quad \text{since } e \bullet (f, f)p = (f, f)p \text{ by Lemma 2.13(ii)} \\ &= (f, (a \bullet f)(f, f)p^{-1}) \quad \text{since } (e, e)p \bullet f = (e, f)p \text{ by Lemma 2.13(ii)} \\ &= (f, (f, f)p^{-1}(aA(e, f))) \quad \text{by Lemma 2.13(i)} \\ &= a\mathbf{A}(e, f)U_f. \end{aligned} \tag{2.69}$$

Similarly we prove the diagram is commutative for other cases $e\mathbf{R}f$ and $e\mathbf{L}f$ also. Hence, by Definition 2.1, (T_p, π_p, U_p) is an extension of S by \mathbf{A} . The proof of Theorem 2.15 is complete. \square

We denote the extension (T_p, π_p, U_p) by $(S, \sigma, p, \mathbf{A})$, and call *the crossed extension of S by \mathbf{A} determined by the crossed pair (σ, p)* .

THEOREM 2.16. *Let $\Psi : S \rightarrow \text{Reg}_E(\mathbf{A})/\text{Inn}_E(\mathbf{A})$ be an abstract kernel and let (σ, p) be a crossed pair, with $(x)\sigma \in (x)\Psi$ for every $x \in S$. Then the abstract kernel of the crossed extension $(S, \sigma, p, \mathbf{A})$ coincides with Ψ .*

Proof. Define $j : S \rightarrow T_p$ by $(x)j = (x, 1_x)$, where 1_x denotes the identity element of \mathbf{A}_x . For each $(x, a) \in T_p$, let

$$(x, a)^* = (x^*, x^* \bullet ((a^{-1} \bullet x^*)(x, x^*)p^{-1}(xx^*, xx^*)p^{-1})). \tag{2.70}$$

Then the proof of Theorem 2.15 shows that $(x, a)^* \in V(x, a)$. Let $\bar{\mu} : T_p \rightarrow \text{Reg}_E(\mathbf{A})$ be the idempotent-separating homomorphism defined by (2.8). Then

$$(x)j\bar{\mu} = [\beta(xj, (xj)^*), \Psi(xj, (xj)^*)], \quad x \in S. \tag{2.71}$$

The proof of the theorem follows once we show that the representative $(\beta(xj, (xj)^*), \Psi(xj, (xj)^*)) : xx^* \rightarrow x^*x$ of $(x)j\bar{\mu}$ and the representative $(\alpha(x), \phi(x)) : xx^* \rightarrow x^*x$ of $(x)\sigma$ in $\mathbf{G}(\mathbf{A})$ are equal. From Remark 2.8(ii) and (2.13) it is clear that

$$\beta(xj, (xj)^*) = \alpha(x) \tag{2.72}$$

we next show that $\Psi(xj, (xj)^*) = \phi(x)$. To prove this we must show that $\Psi_e(xj, (xj)^*) = \phi_e(x) : \mathbf{A}_e \rightarrow \mathbf{A}_{x^*ex}$ for each $e \in \omega(xx^*)$. For this purpose we first make some calculations.

Let $e \in \omega(xx^*)$. Put $d = (x, x^*)p^{-1}(xx^*, xx^*)p^{-1}$. Then, by Lemmas 2.13(ii) and 2.14(i),

$$(x^* \bullet d) \bullet ex = (x^*x \bullet (x^*, ex)p^{-1})(x^* \bullet (x, x^*ex)p^{-1}). \quad (2.73)$$

Therefore, since $x^*x \bullet (x^*, ex)p^{-1} = (x^*, ex)p^{-1}$ by Lemma 2.13(ii),

$$(x^* \bullet d) \bullet ex = (x^*, ex)p^{-1}(x^* \bullet (x, x^*ex)p^{-1}). \quad (2.74)$$

Putting $x = x^*e$, $y = x$, $z = x^*ex$ in (2.48),

$$(x^*ex, x^*ex)p(x^*e, x)p \bullet x^*ex = (x^*e, ex)p(x^*e \bullet (x, x^*ex)p). \quad (2.75)$$

Since $(x^*e, x)p \bullet x^*ex = (x^*ex, x^*ex)p^{-1}(x^*e, x)p(x^*ex, x^*ex)p$ by Lemma 2.13(i) and $x^*e \bullet (x, x^*ex)p = (x^* \bullet (e \bullet (x, x^*ex)p)) = x^* \bullet (x, x^*ex)p$ by Lemma 2.13(ii), the above equation becomes

$$(x^*e, x)p(x^*ex, x^*ex)p = (x^*e, ex)p(x^* \bullet (x, x^*ex)p) \quad (2.76)$$

or

$$(x^* \bullet (x, x^*ex)p^{-1}) = (x^*ex, x^*ex)p^{-1}(x^*e, x)p^{-1}(x^*e, ex)p. \quad (2.77)$$

Since

$$\begin{aligned} (x, x^*e)p \bullet e &= (e, e)p^{-1}(x, x^*e)p(e, e)p \quad \text{by Lemma 2.13(i),} \\ (e, e)p(x, x^*e)p \bullet e &= (x, x^*e)p(x \bullet (x^*e, e)p) \quad \text{by (2.48)} \\ \implies (x, x^*e)p(e, e)p &= (x, x^*e)p(x \bullet (x^*e, e)p) \implies (e, e)p^{-1} = (x \bullet (x^*e, e)p^{-1}) \\ \implies (x^* \bullet ((e, e)p^{-1} \bullet x)) &= x^*x \bullet ((x^*e, e)p^{-1} \bullet x) = (x^*e, e)p^{-1} \bullet x. \end{aligned} \quad (2.78)$$

Also since

$$\begin{aligned} x^*e \bullet (e, x)p &= x^* \bullet (e \bullet (e, x)p) = x^* \bullet (e, x)p \quad \text{by Lemma 2.13(ii),} \\ (x^*e, x)p(x^*e, e)p \bullet x &= (x^*e, ex)p(x^*e \bullet (e, x)p) \quad \text{by (2.48)} \\ \implies x^* \bullet (e, x)p &= (x^*e, ex)p^{-1}(x^*e, x)p(x^*e, e)p \bullet x \\ \implies (x^*e, e)p \bullet x &= (x^*e, x)p^{-1}(x^*e, ex)p(x^* \bullet (e, x)p). \end{aligned} \quad (2.79)$$

For any $a \in \mathbf{A}_e$, by (2.74), (2.77), (2.78), (2.79), and Lemma 2.13(i), it is easy to show

$$(xj)^*(e, (e, e)p^{-1}a)(xj) = (x^*ex, (x^*ex, x^*ex)p^{-1})(x^* \bullet (a \bullet x)). \quad (2.80)$$

But

$$\begin{aligned} x^* \bullet (a \bullet x) &= x^* \bullet (a\mathbf{A}(D(e, e, xx^*))\phi_e(x)\mathbf{A}(L(x^*ex, (ex)^*ex))) \\ &= x^* \bullet (a\phi_e(x)\mathbf{A}(L(x^*ex, (ex)^*ex))) \quad \text{since } D(e, e, xx^*) = 1_e \\ &= a\phi_e(x)\mathbf{A}(L(x^*ex, (ex)^*ex))\mathbf{A}(L((ex)^*ex, x^*ex)) \quad \text{by (2.39)} \\ &= a\phi_e(x) \quad \text{by Lemma 2.13(i).} \end{aligned} \quad (2.81)$$

Hence, $(xj)^*(e, (e, e)p^{-1}a)(xj) = (x^*ex, (x^*ex, x^*ex)p^{-1})(a)\phi_e(x)$. This implies $\Psi_e(xj, (xj)^*) = \phi_e(x)$ for every $e \in \omega(xx^*)$, where we have identified $a \in \mathbf{A}_e$ with $(e, (e, e)p^{-1}a)$ under the isomorphism $(U_p)_e : \mathbf{A}_e \rightarrow \mathbf{A}_e^{\pi p}$. Hence,

$$\Psi(xj, (xj)^*) = \phi(x). \tag{2.82}$$

The result now follows from (2.72) and (2.82). □

LEMMA 2.17 [9, Lemma 4.2]. *Let $(T, \pi, \mathbf{1})$ be an extension of S by \mathbf{A} . Let $j : S \rightarrow T$ be a map such that $j\pi = \mathbf{1}_S$ and let \bullet denote the biaction of S on \mathbf{A} induced by the composite*

$$\sigma : S \xrightarrow{j} T \xrightarrow{\bar{\mu}} \text{Reg}_E(\mathbf{A}), \tag{2.83}$$

where $\bar{\mu}$ is as in (2.16). Then $(xj)a(yj)b = (xj)(yj)(a \bullet y)(x \bullet b)$ for $x, y \in S, a \in \mathbf{A}_x = \mathbf{A}_{x^*x}, b \in \mathbf{A}_y = \mathbf{A}_{y^*y}$.

THEOREM 2.18. *Let $\varepsilon_T = (T, \pi, \mathbf{1})$ be an extension of S by \mathbf{A} with abstract kernel $\Psi : S \rightarrow \text{Reg}_E(\mathbf{A})/\text{Inn}_E(\mathbf{A})$. Let $\sigma : S \rightarrow \text{Reg}_E(\mathbf{A})$ be a map such that $(x)\sigma \in (x)\Psi$ for each $x \in S$. Then ε_T is equivalent to a crossed extension of the form $(S, \sigma, p, \mathbf{A})$ with abstract kernel Ψ .*

Proof. Let $\bar{\mu} : T \rightarrow \text{Reg}_E(\mathbf{A})$ be the idempotent-separating homomorphism defined by (2.16). Using the commutativity of diagram (2.20), it is easy to see that every element in the class $(x)\Psi$ is of the form $[\beta(u, u'), \Psi(u, u')]$ for some $u \in T, u' \in V(u)$, with $u\pi = x$. So there is a map $j : S \rightarrow T$, with $j\pi = \mathbf{1}_S$, such that $\bar{j}\bar{\mu} = \sigma$; in particular $(x)\sigma \in (x)\Psi$ for every $x \in S$. Since $((xj)(yj))\pi = (x)j\pi(y)j\pi = xy = (xy)j\pi$, Lemma 1.6 defines a function $p : S \times S \rightarrow \mathbf{A}$, $(x, y)p \in \mathbf{A}_{xy}$, such that $(xj)(yj) = (xy)j(x, y)p$. This implies, for $x, y \in S, (x)\sigma(y)\sigma = (x)\bar{j}\bar{\mu}(y)\bar{j}\bar{\mu} = ((xj)(yj))\bar{\mu} = (xy)j\bar{\mu}(x, y)p\bar{\mu} = (xy)\sigma(x, y)p\eta$, by (2.19). Again for $x, y, z \in S$, we have by Lemma 2.17, $(xj)((yj)(zj)) = (xj)(yz)j(x, y,z)p = (xj)(yz)j(x \bullet (y, z)p) = (xyz)j(x, yz)p(x \bullet (y, z)p)$, and $((xj)(yj))(zj) = (xy)j(x, y)p(zj) = (xy)j(zj)((x, y)p \bullet z) = (xyz)j(xy, z)p((x, y)p \bullet z)$, where \bullet denotes the biaction of S on \mathbf{A} induced by σ . Since the multiplication in T is associative, by Lemma 1.6,

$$(x, y, z)p((x, y)p \bullet z) = (x, yz)p(x \bullet (y, z)p). \tag{2.84}$$

Thus (σ, p) is a crossed pair.

Next we show that the extension ε_T is equivalent to a crossed extension $(S, \sigma, p, \mathbf{A})$. Define $\theta : T_p \rightarrow T$ by $(x, a)\theta = (xj)a$. Then, by Lemma 2.17, θ is a homomorphism: $((x, a)(y, b))\theta = (xy, (x, y)p(a \bullet y)(x \bullet b))\theta = (xy)j(x, y)p(a \bullet y)(x \bullet b) = (x)j(y)j(a \bullet y)(x \bullet b) = (xj)a(yj)b = (x, a)\theta(y, b)\theta$. From Lemma 1.6, we see that θ is a bijection and therefore an isomorphism. $\theta_\pi = \pi_p$, since $(x, a)\theta_\pi = ((xj)a)\pi = x = (x, a)\pi_p$. Finally the diagram

$$\begin{array}{ccc}
 & \mathbf{A}_e & \\
 U_p \swarrow & & \searrow \\
 \mathbf{A}_e^{\pi p} & \xrightarrow{\theta} & \mathbf{A}_e^\pi
 \end{array} \tag{2.85}$$

is commutative, since $(ej)(ej) = (ej)(e,e)p$ implies $ej = (e,e)p$ and hence for $a \in \mathbf{A}_e$, $(a)U_p\theta = (e,(e,e)p^{-1}a)\theta = (ej)(e,e)p^{-1}a = (e,e)p(e,e)p^{-1}a = a$. Hence, ε_T is equivalent to a crossed extension $(S, \sigma, p, \mathbf{A})$. This completes the proof of Theorem 2.18. \square

Combining Theorems 2.15, 2.16, and 2.18, we obtain a complete description of extensions of S by A which induce the given abstract kernel Ψ in terms of the crossed pairs (σ, p) .

3. Obstructions to extensions

Let S^I be the regular semigroup obtained from S by adjoining an identity element I ($I \notin S$). Extend the map $*$: $S \rightarrow S$ (see (2.36)) to S^I by defining $I^* = I$. Now recall the category $D(S^I)$ [5] as follows. The objects are elements of S^I and morphisms are the triples $\langle u, x, v \rangle : x \rightarrow y$ such that $uxv = y$. The morphism composition is defined by $\langle u, x, v \rangle \langle u', uxv, v' \rangle = \langle u'u, x, vv' \rangle$. Let $\mathbf{F} : D(S^I) \rightarrow D(S^I)$ be the functor defined by $\mathbf{F}(x) = x^*x$ on objects of $D(S^I)$ and $\mathbf{F}\langle u, x, v \rangle = [x^*x, x^*xv y^*y, y^*ux]$ on morphisms $\langle u, x, v \rangle : x \rightarrow y$ of $D(S^I)$ [9]. A functor $\mathbf{G} : D(S^I) \rightarrow \mathbf{Ab}$ is called a $D(S^I)$ -module. For $D(S^I)$ -modules \mathbf{G} and \mathbf{H} , $D(S^I)$ -homomorphism $\phi : \mathbf{G} \rightarrow \mathbf{H}$ is a natural transformation of functors. We denote by $\text{hom}_{D(S^I)}(\mathbf{G}, \mathbf{H})$ the abelian group of all $D(S^I)$ -homomorphisms from \mathbf{G} to \mathbf{H} and by $\text{Mod}(D(S^I))$ the category of $D(S^I)$ -modules and $D(S^I)$ -homomorphisms. $\text{Mod}(D(S^I))$ is an abelian category with enough injectives and projectives. Let $D(S^I)_0$ be the subcategory of $D(S^I)$ defined by the identity morphisms of $D(S^I)$. A $D(S^I)_0$ -set is a functor $\Gamma : D(S^I)_0 \rightarrow \text{Sets}$ from $D(S^I)_0$ to the category of sets, and $D(S^I)_0$ -map is a natural transformation between two $D(S^I)_0$ -sets. A $D(S^I)$ -module (resp., $D(S^I)$ -homomorphism) defines a $D(S^I)_0$ -set (resp., $D(S^I)_0$ -map) in an obvious manner. For more details, refer to [5].

If Γ is a $D(S^I)_0$ -set, then the free $D(S^I)$ -module on Γ is the $D(S^I)$ -module \mathbf{G} such that, for each object y of $D(S^I)$, \mathbf{G}_y is the free abelian group generated by elements of the form $(a, \langle u, x, u' \rangle)$, $a \in \Gamma_x$, $x \in \text{object } D(S^I)$, $uxu' = y$. If $\langle v, y, v' \rangle : y \rightarrow z$ is a morphism of $D(S^I)$, then $\mathbf{G}\langle v, y, v' \rangle : \mathbf{G}_y \rightarrow \mathbf{G}_z$ is defined by

$$(a, \langle u, x, u' \rangle) \mathbf{G}\langle v, y, v' \rangle = (a, \langle vu, x, u'v' \rangle). \tag{3.1}$$

We identify $a \in \Gamma_x$ with $(a, \langle 1, x, 1 \rangle)$ in \mathbf{G}_x . For $n \geq 0$, let x_n be the free $D(S^I)$ -module on the $D(S^I)_0$ -set Γ_n , where, for $n \geq 1$,

$$\Gamma_n(x) = \{ [u_1, \dots, u_n] \in (S^I)^n : u_1u_2 \cdots u_n = x \} \tag{3.2}$$

and, for $n = 0$,

$$\Gamma_0(x) = \begin{cases} \{ [1] \} & \text{if } x = 1, \\ 0 & \text{if } x \neq 1, x \in S^I. \end{cases} \tag{3.3}$$

Now we recall [5, Theorem 2.3]: the complex

$$X \cdots \longrightarrow X_n \xrightarrow{\partial_n} X_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_3} X_2 \xrightarrow{\partial_2} X_1 \xrightarrow{\partial_1} X_0 \xrightarrow{\varepsilon} \mathbf{Z}_{S^I} \longrightarrow 0 \tag{3.4}$$

is called the *standard resolution of \mathbf{Z}_{S^I}* .

Let $\bar{\Gamma}_n(x) = \{[u_1, \dots, u_n] \in \Gamma_n(x) : u_i \neq 1, i = 1, 2, \dots, n\}$, $n \geq 1$, and $\bar{\Gamma}_n = \cup \bar{\Gamma}_n(x)$, $x \in S^I$. Then the $D(S^I)_0$ -set $\bar{\Gamma}_n$ freely generates a $D(S^I)$ -submodule \bar{X}_n of X_n . Put $\bar{X}_0 = X_0$. Define ∂_n as before, putting $[u_1, \dots, u_n] = 0$ whenever one of the u_i is one. Then we obtain another projective resolution

$$\bar{X} \cdots \longrightarrow \bar{X}_n \xrightarrow{\partial_n} \bar{X}_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_3} \bar{X}_2 \xrightarrow{\partial_2} \bar{X}_1 \xrightarrow{\partial_1} \bar{X}_0 \xrightarrow{\epsilon} \mathbf{Z}_{S^I} \longrightarrow 0 \tag{3.5}$$

of \mathbf{Z}_{S^I} , called *the normalised standard resolution of \mathbf{Z}_{S^I}* .

Let $\mathbf{G} \in \text{Mod}(\mathbf{D}(S^I))$ and let

$$\begin{aligned} \text{hom}_{D(S^I)}(\bar{X}, \mathbf{G}) : 0 \longrightarrow \text{hom}_{D(S^I)}(\bar{X}_0, \mathbf{G}) \xrightarrow{\partial_1^*} \text{hom}_{D(S^I)}(\bar{X}_1, \mathbf{G}) \xrightarrow{\partial_2^*} \cdots \\ \xrightarrow{\partial_{n-1}^*} \text{hom}_{D(S^I)}(\bar{X}_{n-1}, \mathbf{G}) \xrightarrow{\partial_n^*} \text{hom}_{D(S^I)}(\bar{X}_n, \mathbf{G}) \xrightarrow{\partial_{n+1}^*} \cdots \end{aligned} \tag{3.6}$$

Definition 3.1. The *n*th cohomology group of S^I with coefficients in \mathbf{G} , denoted by $\mathbf{H}^n(S^I, \mathbf{G})$, is defined by

$$\mathbf{H}^n(S^I, \mathbf{G}) = \mathbf{H}^n[\text{hom}_{D(S^I)}(\bar{X}, \mathbf{G})] = \text{Ker } \partial_{n+1}^* / \text{Im } \partial_n^*. \tag{3.7}$$

The elements of $\text{hom}_{D(S^I)}(\bar{X}, \mathbf{G})$ are called (*normalized*) *n-cochains*. The elements of $\text{Ker } \partial_{n+1}^*$ are called (*normalized*) *n-cocycles* and the elements of $\text{Im } \partial_n^*$ are called (*normalized*) *n-coboundaries*. Two *n-cocycles* $k_1, k_2 \in \text{Ker } \partial_{n+1}^*$ are called *cohomologous* if they differ by a coboundary.

Let $\mathbf{A} : \underline{C}(E) \rightarrow \mathbf{GR}$ be a group *E*-diagram that factors through $\mathbf{D}(B(E))$ and let $\mathbf{Z}(\mathbf{A})$ be the centre of \mathbf{A} . For each $x \in S$, let $\bar{\mathbf{Z}}(\mathbf{A})_x = \mathbf{Z}(\mathbf{A})_{x^*x}$ and let

$$\bar{\mathbf{Z}}(\mathbf{A}) = \bigcup_{x \in S} \bar{\mathbf{Z}}(\mathbf{A})_x \tag{3.8}$$

be the disjoint union of $\bar{\mathbf{Z}}(\mathbf{A})_x$'s. Remark that $\bar{\mathbf{Z}}(\mathbf{A})_x$ is contained in the centre of $\bar{\mathbf{A}}_x$, where as in the previous section $\bar{\mathbf{A}} = \bigcup_{x \in S} \bar{\mathbf{A}}_x$ with $\mathbf{A}_x = \bar{\mathbf{A}}_{x^*x}$.

Suppose $\Psi : S \rightarrow \text{Reg}_E(\mathbf{A})/\text{Inn}_E(\mathbf{A})$ is an abstract kernel. Then the composite $S \xrightarrow{\Psi} (\text{Reg}_E(\mathbf{A})/\text{Inn}_E(\mathbf{A})) \xrightarrow{V} \text{Reg}_E(\mathbf{Z}(\mathbf{A}))$ is an idempotent-separating homomorphism. Since \mathbf{A} and hence $\mathbf{Z}(\mathbf{A}) : \underline{C}(E) \rightarrow \mathbf{Ab}$ factors through $D(B(E))$, by Remark 2.9, Ψv induces a functor $\check{\mathbf{Z}}(\mathbf{A}) = \check{\mathbf{Z}}(\mathbf{A})_{\Psi v} : \mathbf{D}(S) \rightarrow \mathbf{Ab}$. Let $\check{\mathbf{Z}}(\mathbf{A})^0 : \mathbf{D}(S^I) \rightarrow \mathbf{Ab}$ be the extension of $\check{\mathbf{Z}}(\mathbf{A})$ such that $\check{\mathbf{Z}}(\mathbf{A})_I^0 = \{0\}$ and let $\check{\mathbf{Z}}(\mathbf{A})^0 \mathbf{F}$ be the composite $D(S^I) \xrightarrow{\mathbf{F}} \mathbf{D}(S^I) \xrightarrow{\check{\mathbf{Z}}(\mathbf{A})^0} \mathbf{Ab}$. In this section, we associate with the abstract kernel Ψ a 3-dimensional cohomology class $[k] \in \mathbf{H}^3(S^I, \check{\mathbf{Z}}(\mathbf{A})^0 \mathbf{F})$ and show that Ψ admits an extension if and only if $[k] = 0$. We also show that if Ψ has an extension, then the set of all equivalence classes of extensions of S by \mathbf{A} is in bijective correspondence with the set $\mathbf{H}^2(S^I, \check{\mathbf{Z}}(\mathbf{A})^0 \mathbf{F})$.

Let $\sigma : S \rightarrow \text{Reg}_E(\mathbf{A})$ be any map such that $(x)\sigma \in (x)\Psi$. As before for each $x \in S$, let $(\alpha(x), \phi(x)) : xx^* \rightarrow x^*x$ denote the unique representative of $(x)\sigma$ in $\mathbf{G}(\mathbf{A})$ with domain xx^* and range x^*x and let $(\alpha(x), \bar{\phi}(x)) : xx^* \rightarrow x^*x$ denote the element of $\mathbf{G}(\mathbf{Z}(\mathbf{A}))$ determined by $(\alpha(x), \phi(x))$ (see (2.13)) so that $(x)\Psi v = (x)\sigma u = [\alpha(x), \phi(x)]$. The biaction of S on \mathbf{A} defined by σ induces by restriction a biaction of S on $\overline{\mathbf{Z}(\mathbf{A})}$ which coincides with the one induced by the composite $\Psi v = \sigma u : S \rightarrow \text{Reg}_E(\mathbf{A}) \rightarrow \text{Reg}_E(\mathbf{Z}(\mathbf{A}))$. In particular, the induced biaction of S on $\overline{\mathbf{Z}(\mathbf{A})}$ is independent of the chosen σ . We next see the relation between this biaction and the functor $\check{\mathbf{Z}}(\mathbf{A})^0\mathbf{F} : D(S^I) \rightarrow \mathbf{Ab}$. Let $x \in S$, $a \in \overline{\mathbf{Z}(\mathbf{A})}_y = \mathbf{Z}(\mathbf{A})_{y^*y} = (\check{\mathbf{Z}}(\mathbf{A})^0\mathbf{F})_y$. Then by (2.23) and (2.39) we have

$$\begin{aligned}
 a(\check{\mathbf{Z}}(\mathbf{A})^0\mathbf{F}(x, y, I)) &= a\check{\mathbf{Z}}(\mathbf{A})^0[y^*y, y^*y(xy)^*xy, (xy)^*xy] \\
 &= a\check{\mathbf{Z}}(\mathbf{A})[y^*y, y^*y(xy)^*xy, (xy)^*xy] \\
 &= a\mathbf{Z}(\mathbf{A})(L(y^*y, (xy)^*xy)) = x \bullet a, \\
 a(\check{\mathbf{Z}}(\mathbf{A})^0\mathbf{F}(I, y, x)) &= a\check{\mathbf{Z}}(\mathbf{A})^0[y^*y, y^*yx, (yx)^*y] \\
 &= a\check{\mathbf{Z}}(\mathbf{A})[y^*y, y^*yx, (yx)^*y] \\
 &= a\check{\mathbf{Z}}(\mathbf{A})[y^*y, y^*yx, (yx)^*yh] \quad \text{by (1.2)} \\
 &= a\check{\mathbf{Z}}(\mathbf{A})\{[y^*y, y^*yxx^*, h] \\
 &\quad \times [hxx^*, hx, x^*hxx^*][x^*hx, x^*hx, (yx)^*yx]\} \\
 &= a\mathbf{Z}(\mathbf{A})(D(h, y^*y, xx^*))\check{\mathbf{Z}}(\mathbf{A})[hxx^*, hx, x^*hxx^*] \\
 &\quad \times \mathbf{Z}(\mathbf{A})(L(x^*hx, (yx)^*yx)) \quad \text{by (2.23)} \\
 &= a\mathbf{Z}(\mathbf{A})(D(h, y^*y, xx^*))\bar{\phi}_{hxx^*}(hx)\mathbf{Z}(\mathbf{A})(L(x^*hx, (yx)^*yx)) \\
 &\quad \text{by Remark 2.9} \\
 &= a\mathbf{Z}(\mathbf{A})(D(h, y^*y, xx^*))\bar{\phi}_{hxx^*}(x)\mathbf{Z}(\mathbf{A})(L(x^*hx, (yx)^*yx)) \\
 &= a \bullet x \quad \text{by (2.40),}
 \end{aligned} \tag{3.9}$$

where $h \in S(y^*y, xx^*)$, and the components $\bar{\phi}_{hxx^*}(hx)$ of $\bar{\phi}(hx)$ and $\bar{\phi}_{hxx^*}(x)$ of $\bar{\phi}(x)$ are equal since $[\alpha(hx), \bar{\phi}(hx)] = (hx)\Psi v = (hxx^*)\Psi v(x)\Psi v = [1_{hxx^*}, \mathbf{1}_{hxx^*}][\alpha(x), \bar{\phi}(x)] = [hxx^* * (\alpha(x), \phi(x))]$. Thus we have $x \bullet a = a(\check{\mathbf{Z}}(\mathbf{A})^0\mathbf{F}(x, y, I))$ and $a \bullet x = a(\check{\mathbf{Z}}(\mathbf{A})^0\mathbf{F}(I, y, x))$.

Next we describe the cohomology groups. Consider the normalized standard resolution (3.5). Since the $D(S^I)$ -module \overline{X}_n 's are free on $\overline{\Gamma}_n$'s and since $(\check{\mathbf{Z}}(\mathbf{A})^0\mathbf{F})_I = \{0\}$, we have

$$\begin{aligned}
 \text{hom}_{D(S^I)}(\overline{X}_n, \check{\mathbf{Z}}(\mathbf{A})^0\mathbf{F}) &= \text{hom}_{D(S^I)_0}(\overline{\Gamma}_n, \check{\mathbf{Z}}(\mathbf{A})^0\mathbf{F}) \\
 &= \left\{ \alpha : S \times S \times \cdots \times S \longrightarrow \overline{\mathbf{Z}(\mathbf{A})} : (x_1, x_2, \dots, x_n) \in \overline{\mathbf{Z}(\mathbf{A})}_{x_1x_2 \cdots x_n} \right\}.
 \end{aligned} \tag{3.10}$$

Hence, we may regard an n -cochain as a map $\alpha : S \times S \times \cdots \times S \rightarrow \overline{\mathbf{Z}(\mathbf{A})}$, with $(x_1, x_2, \dots, x_n) \in \overline{\mathbf{Z}(\mathbf{A})}_{x_1 x_2 \cdots x_n}$. The coboundary $\partial_n^* \alpha$ of an $n - 1$ cochain α is given by the formula

$$\begin{aligned} (x_1, x_2, \dots, x_n) \partial_n^* \alpha &= (x_2, x_3, \dots, x_n) \alpha \check{\mathbf{Z}}(\mathbf{A})^0 \mathbf{F} \langle x_1, x_2, \dots, x_n, I \rangle \\ &\quad + \sum_{i=1}^{n-1} (-1)^i (x_1, x_2, \dots, x_i x_{i+1}, \dots, x_n) \alpha \\ &\quad + (-1)^n (x_1, x_2, \dots, x_{n-1}) \alpha \check{\mathbf{Z}}(\mathbf{A})^0 \mathbf{F} \langle I, x_1, x_2, \dots, x_{n-1}, x_n \rangle \quad (3.11) \\ &= x_1 \bullet (x_2, x_3, \dots, x_n) \alpha + \sum_{i=1}^{n-1} (-1)^i (x_1, x_2, \dots, x_i x_{i+1}, \dots, x_n) \alpha \\ &\quad + (-1)^n (x_1, x_2, \dots, x_{n-1}) \alpha \bullet x_n. \end{aligned}$$

From now on we write the group operation as multiplication. Note that a 2-cochain $\alpha : S \times S \rightarrow \overline{\mathbf{Z}(\mathbf{A})}$, $(x, y) \in \overline{\mathbf{Z}(\mathbf{A})}_{xy}$, is a 2-cocycle if

$$(xy, z) \alpha ((x, y) \alpha \bullet z) = (x, yz) \alpha (x \bullet (y, z) \alpha) \quad (3.12)$$

for all $x, y, z \in S$; α is a coboundary if and only if there exists a 1-cochain $\beta : S \rightarrow \overline{\mathbf{Z}(\mathbf{A})}$, $(x) \beta \in \overline{\mathbf{Z}(\mathbf{A})}_x$, such that

$$(x, y) \alpha = (x \bullet (y) \beta) (xy) \beta^{-1} ((x) \beta \bullet y) \quad (3.13)$$

for all $x, y \in S$. Similarly a 3-cocycle k is a map $k : S \times S \times S \rightarrow \overline{\mathbf{Z}(\mathbf{A})}$, $(x, y, z) k \in \overline{\mathbf{Z}(\mathbf{A})}_{xyz}$, such that

$$(xy, z, t) k (x, y, zt) k = ((x, y, z) k \bullet t) (x, yz, t) k (x \bullet (y, z, t) k) \quad (3.14)$$

for all $x, y, z, t \in S$; k is a coboundary if and only if there exists a 2-cochain $\alpha : S \times S \rightarrow \overline{\mathbf{Z}(\mathbf{A})}$, $(x, y) \in \overline{\mathbf{Z}(\mathbf{A})}_{xy}$, such that

$$(x, y, z) k = (x \bullet (y, z) \alpha) (xy, z) \alpha^{-1} (x, yz) \alpha ((x, y) \alpha \bullet z)^{-1} \quad (3.15)$$

for all $x, y, z \in S$. For $n = 2, 3$, let $\mathbf{Z}^n(S^I, \check{\mathbf{Z}}(\mathbf{A})^0 \mathbf{F})$ denote the abelian group of all n -cocycles and let $\mathbf{B}^n(S^I, \check{\mathbf{Z}}(\mathbf{A})^0 \mathbf{F}) \subseteq \mathbf{Z}^n(S^I, \check{\mathbf{Z}}(\mathbf{A})^0 \mathbf{F})$ be the subgroup of all coboundaries. Then

$$\mathbf{H}^n(S^I, \check{\mathbf{Z}}(\mathbf{A})^0 \mathbf{F}) = \frac{\mathbf{Z}^n(S^I, \check{\mathbf{Z}}(\mathbf{A})^0 \mathbf{F})}{\mathbf{B}^n(S^I, \check{\mathbf{Z}}(\mathbf{A})^0 \mathbf{F})}. \quad (3.16)$$

Now we proceed to show that the abstract kernel $\Psi : S \rightarrow \text{Reg}_E(\mathbf{A})/\text{Inn}_E(\mathbf{A})$ defines an element in the cohomology group $\mathbf{H}^3(S^I, \check{\mathbf{Z}}(\mathbf{A})^0 \mathbf{F})$, the vanishing of which is necessary and sufficient for the existence of extensions of Ψ .

We fix a map $\sigma : S \rightarrow \text{Reg}_E(\mathbf{A})$ such that $(x)\sigma \in (x)\Psi$ for all $x \in S$. Let \bullet denote the biaction of S on \mathbf{A} induced by σ . As before we denote by $(\alpha(x), \phi(x)) : xx^* \rightarrow x^*x$ the unique representative of $(x)\sigma$ in $\mathbf{G}(\mathbf{A})$ with domain xx^* and range x^*x . Since $(x\sigma)(y\sigma)$ and $(xy)\sigma$ both belong to the same class $(xy)\Psi$, we can choose a function $p : S \times S \rightarrow \mathbf{A}$, $(x, y)p \in \mathbf{A}_{xy}$, such that

$$(x)\sigma(y)\sigma = (xy)\sigma((x, y)p)\eta, \tag{3.17}$$

where $\eta : \mathbf{A}_{xy} = \mathbf{A}_{(xy)^*xy} \rightarrow \text{Reg}_E(\mathbf{A})$ is as before.

Before proceeding further, let us first prove the following.

LEMMA 3.2. For $a \in \mathbf{A}_x$, $b \in \mathbf{A}_y$,

$$(x\sigma)(a)\eta(y\sigma)(b)\eta = (xy)\sigma((x, y)p(a \bullet y)(x \bullet b))\eta. \tag{3.18}$$

Proof. Consider the diagram

$$\begin{array}{ccccccc}
 xhx^* & \xrightarrow{(\alpha, \phi)} & x^*xh & \xrightarrow{(\alpha^{c_1}, \phi^{c_1})} & hyy^* & \xrightarrow{(\beta, \Psi)} & y^*hy & \xrightarrow{(1_{y^*hy}, \eta^{(aG)(bH)})} & y^*hy \\
 \downarrow (\alpha^2, \phi^2) & & & & & & \downarrow (\alpha^3, \phi^3) & & \downarrow (\alpha^3, \phi^3) \\
 xy(xy)^* & \xrightarrow{(\alpha(xy), \phi(xy))(1_{(xy)^*xy}, \eta^{(x,y)p})} & & & (xy)^*xy & \xrightarrow{(1_{(xy)^*xy}, \eta^{(a \bullet y)(x \bullet b)})} & & & (xy)^*xy
 \end{array} \tag{3.19}$$

where

$$\begin{aligned}
 h \in S(x^*x, yy^*), \quad c_1 = c(x^*xh, h, hyy^*), \quad c_2 = c(xhx^*, xy(xy)^*), \\
 c_3 = c(y^*hy, (xy)^*xy), \\
 \varepsilon(c_1) = (\alpha^{c_1}, \phi^{c_1}), \quad \varepsilon(c_2) = (\alpha^{c_2}, \phi^{c_2}), \quad \varepsilon(c_3) = (\alpha^{c_3}, \phi^{c_3}), \tag{3.20}
 \end{aligned}$$

$$\begin{aligned}
 (\alpha, \phi) &= (\alpha(x), \phi(x)) * x^*xh : xhx^* \rightarrow x^*xh, \\
 (\beta, \Psi) &= hyy^* * (\alpha(y), \phi(y)) : hyy^* \rightarrow y^*hy,
 \end{aligned}$$

$$\begin{aligned}
 H &= \mathbf{A}(L(y^*y, y^*hy)), \\
 G &= \mathbf{A}(L(x^*x, x^*xh))(\phi^{c_1})_{x^*xh} \Psi_{hyy^*} \\
 &= \mathbf{A}(L(x^*x, x^*xh))\mathbf{A}(D(h, x^*xh, yy^*))\Psi_{hyy^*} \\
 &= \mathbf{A}(D(h, x^*x, yy^*))\Psi_{hyy^*} \quad \text{by Lemma 2.10(ii),}
 \end{aligned} \tag{3.21}$$

$$\begin{aligned}
 a \bullet y &= (a)G\mathbf{A}(L(y^*hy, (xy)^*xy)) = (a)G(\phi^{c_3})_{y^*hy} \quad \text{by [9, (1.8)],} \\
 x \bullet b &= (b)\mathbf{A}(L(y^*y, (xy)^*xy)) = (b)\mathbf{A}(L(y^*hy, (xy)^*xy)) = (b)H(\phi^{c_3})_{y^*hy} \\
 &\quad \text{by [10, (1.8)].}
 \end{aligned} \tag{3.22}$$

Since $(x\sigma)(y\sigma) = (xy)\sigma((x, y)p)\eta$, the first rectangle is commutative. The second rectangle is also commutative, since for $d \in \mathbf{A}_{y^*hy}$,

$$\begin{aligned}
 d(\eta^{(aG)(bH)})_{y^*hy}(\phi^{c_3})_{y^*hy} &= (((aG)(bH))^{-1}d(aG)(bH))(\phi^{c_3})_{y^*hy} \\
 &= ((aG)(bH))^{-1}(\phi^{c_3})_{y^*hy}d(\phi^{c_3})_{y^*hy}((aG)(bH))(\phi^{c_3})_{y^*hy} \tag{3.23} \\
 &= ((a \bullet y)(x \bullet b))^{-1}(d\phi^{c_3})_{y^*hy}((a \bullet y)(x \bullet b)) \\
 &= (d)(\phi^{c_3})_{y^*hy}(\eta^{(a \bullet y)(x \bullet b)})(xy)^*xy \\
 &= (d)(\phi^{c_3}(\underline{C}(\alpha^{c_3})\eta^{(a \bullet y)(x \bullet b)}))_{y^*hy}.
 \end{aligned}$$

Hence, the outer diagram is commutative. Now

$$\begin{aligned}
 (x\sigma)(a)\eta(y)\sigma(b)\eta &= [\alpha(x), \phi(x)][1_{x^*x}, \eta^a][\alpha(y), \phi(y)][1_{y^*y}, \eta^b] \\
 &= [(\alpha, \phi)((1_{x^*x}, \eta^a) * x^*xh)\varepsilon(c_1)(\beta, \Psi)(y^*hy^*(1_{y^*y}, \eta^b))] \text{ by (1.10)} \\
 &= [(\alpha, \phi)\varepsilon(c_1)(\beta, \Psi)(1_{y^*hy}, \eta^{aG})(1_{y^*hy}, \eta^{bH})] \\
 &= [(\alpha(xy), \phi(xy))(1_{(xy)^*xy}, \eta^{(x,y)p(a \bullet y)(x \bullet b)})] \text{ by the diagram} \\
 &= [\alpha(xy), \phi(xy)][1_{(xy)^*xy}, \eta^{(x,y)p(a \bullet y)(x \bullet b)}] \\
 &= (xy)\sigma((x, y)p(a \bullet y)(x \bullet b))\eta. \tag{3.24}
 \end{aligned}$$

Hence, the proof of the lemma is complete. □

Let σ and p be as before. Using (3.17) and Lemma 3.2, we get

$$\begin{aligned}
 ((x\sigma)(y\sigma))(z\sigma) &= (xy)\sigma((x, y)p)\eta(z\sigma) = (xyz)\sigma((x, y)p \bullet z)\eta, \\
 (x\sigma)((y\sigma)(z\sigma)) &= (x\sigma)(yz)\sigma((y, z)p)\eta = (xyz)\sigma((x, yz)p(x \bullet (y, z)p))\eta. \tag{3.25}
 \end{aligned}$$

Since the multiplication in $\text{Reg}_E(\mathbf{A})$ is associative, by Lemma 1.6,

$$((xy, z)p((x, y)p \bullet z))\eta = ((x, yz)p(x \bullet (y, z)p))\eta. \tag{3.26}$$

The exactness of the sequence in Proposition 2.5 gives us a 3-cochain $k : S \times S \times S \rightarrow \overline{\mathbf{Z}(\mathbf{A})}$ such that

$$(xy, z)p((x, y)p \bullet z) = (x, yz)p(x \bullet (y, z)p)(x, y, z)k \tag{3.27}$$

for all $x, y, z \in S$.

LEMMA 3.3. *The map $k : S \times S \times S \rightarrow \overline{\mathbf{Z}(\mathbf{A})}$ is a 3-cocycle.*

Proof. We must show that k satisfies (3.14). Let $x, y, z, t \in S$. Following [11], it is easy to calculate the expression

$$L = (xyz, t)p[(xy, z)p((x, y)p \bullet z)] \bullet t \tag{3.28}$$

in two ways. In the first way using (3.27) and Lemma 2.14, we easily get

$$L = (x, yzt)p(x \bullet (y, zt)p)(xy \bullet (z, t)p)(x \bullet (y, z, t)k)(x, yz, t)k((x, y, z)k \bullet t). \quad (3.29)$$

In the second way also using Lemma 2.14(iii) to the term $((x, y)p \bullet z) \bullet t$, we get

$$L = (xyz, t)p((xy, z)p \bullet t)(xy \bullet (z, t)p)^{-1}((x, y)p \bullet zt)(xy \bullet (z, t)p). \quad (3.30)$$

Using (3.27) to the first two terms and since $(xy, z, t)k \in \overline{\mathbf{Z}(\mathbf{A})}_{xyz, t}$, $(x, y, zt)k \in \overline{\mathbf{Z}(\mathbf{A})}_{x, yzt}$, we finally get

$$L = (x, yzt)p(x \bullet (y, zt)p)(xy \bullet (z, t)p)(x, y, zt)k(xy, z, t)k. \quad (3.31)$$

Comparison gives

$$(xy, z, t)k(x, y, zt)k = ((x, y, z)k \bullet t)(x, yz, t)k(x \bullet (y, z, t)k). \quad (3.32)$$

Hence, by (3.14) k is a 3-cocycle. □

Definition 3.4. The cocycle k satisfying (3.27) is called an *obstruction of the abstract kernel* $\Psi : S \rightarrow \text{Reg}_E(\mathbf{A})/\text{Inn}_E(\mathbf{A})$. The following lemma shows that the cohomology class defined by k is independent of chosen σ and p .

LEMMA 3.5. (i) For a given σ , a change in the choice of p in (3.17) replaces k by a cohomologous cocycle. By suitably changing the choice of p , k may be replaced by any cohomologous cocycle.

(ii) A change in the choice of σ may be followed by a suitable new selection of p so as to leave the obstruction cocycle k unchanged.

Proof. (i) Suppose p' is another choice of p and let k' be the corresponding 3-cocycle so that

$$(xy, z)p'((x, y)p' \bullet z) = (x, yz)p'(x \bullet (y, z)p')(x, y, z)k' \quad (3.33)$$

for all $x, y, z \in S$. We will show that k, k' are cohomologous. Since p and p' satisfy (3.17), by Lemma 1.6, $((x, y)p)\eta = ((x, y)p')\eta$. So the exactness of the sequence in Proposition 2.5 gives rise to a 2-cochain $\tau : S \times S \rightarrow \overline{\mathbf{Z}(\mathbf{A})}$ such that

$$(x, y)p' = (x, y)p(x, y)\tau. \quad (3.34)$$

Substituting (3.34) in (3.33) and using (3.17), we get

$$(x, y, z)kk'^{-1} = (x, y, z)k(x, y, z)k'^{-1} = (x \bullet (y, z)\tau)(xy, z)\tau^{-1}(x, yz)\tau((x, y)\tau \bullet z)^{-1} \quad (3.35)$$

for all $x, y, z \in S$. Thus by (3.15) k and k' are cohomologous. To prove the second statement, take any 3-cocycle k' that is cohomologous to k . Then there is a 2-cochain $\tau : S \times S \rightarrow \overline{\mathbf{Z}(\mathbf{A})}$ such that (3.35) holds. If we put $(x, y)p' = (x, y)p(x, y)\tau$, $x, y \in S$, then p' satisfies (3.17) and (3.33), and so k' is the 3-cocycle defined by p' .

(ii) Let $\sigma' : S \rightarrow \text{Reg}_E(\mathbf{A})$ be another map such that $(x)\sigma' \in (x)\Psi$ for all $x \in S$, and let \circ denote the biaction of S on \mathbf{A} induced by σ' . Then by Lemma 2.11(ii) there exists a map $\beta : S \rightarrow \mathbf{A}$, $(x)\beta \in A_x$, such that $(x)\sigma' = (x)\sigma((x)\beta)\eta$ for all $x \in S$. This implies by Lemma 3.2

$$(x\sigma')(y\sigma') = (xy)\sigma'((xy)\beta^{-1}(x,y)p((x)\beta \bullet y)(x \bullet (y)\beta))\eta. \tag{3.36}$$

Put $(x,y)p' = (xy)\beta^{-1}(x,y)p((x)\beta \bullet y)(x \bullet (y)\beta)\eta$. Then, by Lemma 2.11(i) and (ii),

$$(x,y)p' = (xy)\beta^{-1}(x,y)p(x \circ (y)\beta)((x)\beta \circ y). \tag{3.37}$$

By Lemma 2.14, (3.37), and by the relation $x \circ (y \circ (z)\beta) = xy \circ (z)\beta$, we have

$$(xyz)\beta(xy,z)p'((x,y)p' \circ z) = (xyz)\beta(x,yz)p'(x \circ (y,z)p')(x,y,z)k. \tag{3.38}$$

Hence,

$$(x,y,z)p'((x,y)p' \circ z) = (x,yz)p'(x \circ (y,z)p')(x,y,z)k. \tag{3.39}$$

Thus the obstruction cocycle determined by p' coincides with k . □

From Lemmas 3.3 and 3.5, we obtain the first part of the following.

THEOREM 3.6. *Let $\Psi : S \rightarrow \text{Reg}_E(\mathbf{A})/\text{Inn}_E(\mathbf{A})$ be an abstract kernel. Then Ψ defines a well-defined element $[k]$ of $\mathbf{H}^3(S^l, \check{\mathbf{Z}}(\mathbf{A})^0\mathbf{F})$. Further, Ψ has an extension of S by \mathbf{A} if and only if $[k] = 0$.*

Proof. If Ψ has an extension, then by Theorem 2.18 there is an extension of the form $(S, \sigma, p, \mathbf{A})$ with abstract kernel Ψ and crossed pair (σ, p) . Then, since (σ, p) satisfies (2.48), it is clear from (3.27) that $[k] = 0$. Conversely, suppose $[k] = 0$. In view of Lemma 3.5(i), we can assume without loss of generality that $k = 0$, the zero 3-cocycle. Then (σ, p) is a crossed pair by (3.17) and (3.27), and the crossed extension $(S, \sigma, p, \mathbf{A})$ is an extension of S by \mathbf{A} with abstract kernel Ψ by Theorems 2.15 and 2.16. □

THEOREM 3.7. *Let $(S, \sigma, p, \mathbf{A})$ and $(S, \sigma, q, \mathbf{A})$ be two crossed extensions of S by \mathbf{A} with abstract kernel Ψ . Then $(S, \sigma, p, \mathbf{A})$ is equivalent to $(S, \sigma, q, \mathbf{A})$ if and only if there exists an 1-cochain $\beta : S \rightarrow \overline{\mathbf{Z}}(\mathbf{A})$ such that*

$$(x,y)p(x,y)q^{-1} = ((x)\beta \bullet y)(x \bullet (y)\beta)(xy)\beta^{-1} \tag{3.40}$$

for all $x, y \in S$.

Proof. Suppose $(S, \sigma, p, \mathbf{A})$ and $(S, \sigma, q, \mathbf{A})$ are equivalent extensions and let $\theta : T_p \rightarrow T_q$ be an isomorphism such that $\theta\pi_q = \pi_p$ and $(a)U_p\theta = (a)U_q$, $a \in \mathbf{A}_e$, $e \in E$. Define maps $j_1 : S \rightarrow T_p$ and $j_2 : S \rightarrow T_q$ by $(x)j_1 = (x, 1_x)$; $(x)j_2 = (x, 1_x)$, where 1_x denotes the identity element of \mathbf{A}_x for all $x \in S$. Then $j_1\pi_p = 1_S = j_2\pi_q$. For $x, y \in S$ and by Lemma 2.13(i) and (ii), we can easily show that $(xy)j_1((x,y)p)U_p = (xj_1)(yj_1)$. That is, $(xj_1)(yj_1) = (xy)j_1((x,y)p)U_p$. Similarly $(xj_2)(yj_2) = (xy)j_2((x,y)q)U_q$. The proof of Theorem 2.16 gives $(x)j_1\bar{\mu}_1 = (x)\sigma(x)j_2\bar{\mu}_2$ for all $x \in S$, where $\bar{\mu}_1 : T_p \rightarrow \text{Reg}_E(\mathbf{A})$ and $\bar{\mu}_2 : T_q \rightarrow \text{Reg}_E(\mathbf{A})$

are defined by (2.16). If we denote the composite $j_1\theta : S \rightarrow T_q$ by j , then $j\pi_q = 1_S$, $(xj)(yj) = (xy)j((x, y)p)U_q$, $(x)\sigma = (x)j\bar{\mu}_2$ for all $x, y \in S$. Since $j\pi_q = 1_S = j_2\pi_q$, by Lemma 1.6 there exists a map $\beta : S \rightarrow \mathbf{A}$, $(x)\beta \in \mathbf{A}_x$, such that

$$(x)j = (x)j_2((x)\beta)U_q, \tag{3.41}$$

and so $(x)\sigma = (x)j\bar{\mu}_2 = (x)j_2\bar{\mu}_2((x)\beta)U_q\bar{\mu}_2 = (x)\sigma((x)\beta)\eta$ for all $x \in S$. Then, by Lemma 1.6, $(x)\beta \in \text{Ker } \eta$ or $((x)\beta)\eta = (1_x)\eta$ (where 1_x is the identity element of \mathbf{A}_x) and therefore by Proposition 2.5, $(x)\beta \in \overline{\mathbf{Z}(\mathbf{A})}_x$ for all $x \in S$. Thus β is a 1-cochain. By using Lemma 2.17 and $((x)\beta \bullet y)(x \bullet (y)\beta) \in \overline{\mathbf{Z}(\mathbf{A})}_{xy}$, we easily derive

$$(xy)j = (xy)j[(xy)\beta^{-1}(x, y)q(x, y)p^{-1}((x)\beta \bullet y)(x \bullet (y)\beta)]U_q. \tag{3.42}$$

Then, by Lemma 1.6, $(xy)\beta^{-1}(x, y)q(x, y)p^{-1}((x)\beta \bullet y)(x \bullet (y)\beta) = 1_{xy}$ or, since β takes values in $\overline{\mathbf{Z}(\mathbf{A})}$,

$$(x, y)p(x, y)q^{-1} = ((x)\beta \bullet y)(x \bullet (y)\beta)(xy)\beta^{-1}. \tag{3.43}$$

Conversely, let $\beta : S \rightarrow \overline{\mathbf{Z}(\mathbf{A})}$ be a 1-cochain such that (3.40) holds. This implies, in particular, $(x)\beta$ commutes with every element of \mathbf{A}_x , $x \in S$. Define a map $\theta : T_p \rightarrow T_q$ by $(x, a)\theta = (x, a(x)\beta)$ for all $(x, a) \in T_p$. Then clearly $\theta\pi_q = \pi_p$. Moreover, for $e \in E(S)$ and $a \in \mathbf{A}_e$, by (2.65) we get $(a)U_p\theta = (e, (e, e)p^{-1}a)\theta = (e, (e, e)p^{-1}(e)\beta a) = (e, (e, e)q^{-1}a) = (a)U_q$ since (3.40) implies $(e, e)q^{-1} = (e, e)p^{-1}(e)\beta \bullet e$, and by Lemma 2.13(i), $(e)\beta \bullet e = (e)\beta$. Using (3.40) we can easily verify θ is an isomorphism. Hence, $(S, \sigma, p, \mathbf{A})$ and $(S, \sigma, q, \mathbf{A})$ are equivalent. \square

THEOREM 3.8. *If the abstract kernel $\Psi : S \rightarrow \text{Reg}_E(\mathbf{A})/\text{Inn}_E(\mathbf{A})$ has an extension, then the set $\varepsilon(S, \mathbf{A})$ of equivalence classes of extensions of S by \mathbf{A} with abstract kernel Ψ is in one-to-one correspondence with the set $\mathbf{H}^2(S^l, \check{\mathbf{Z}}(\mathbf{A})^0\mathbf{F})$.*

Proof. Since Ψ admits an extension of S by \mathbf{A} , by Theorem 2.18, there is an extension of the form $(S, \sigma, p, \mathbf{A})$ with abstract kernel Ψ . Keep σ fixed. Let $\alpha : S \times S \rightarrow \overline{\mathbf{Z}(\mathbf{A})}$ be a 2-cocycle so that $(xy, z)\alpha((x, y)\alpha \bullet z) = (x, yz)\alpha(x \bullet (y, z)\alpha)$ for all $x, y, z \in S$. Define $p\alpha : S \times S \rightarrow \mathbf{A}$ by $(x, y)p\alpha = (x, y)p(x, y)\alpha$. Then $(\sigma, p\alpha)$ is a crossed pair and hence defines a crossed extension $(S, \sigma, p\alpha, \mathbf{A})$ with abstract kernel Ψ . If α' is another 2-cocycle, then

$$\begin{aligned} (x, y)\alpha^{-1}(x, y)\alpha' &= (x, y)\alpha^{-1}(x, y)p^{-1}(x, y)p(x, y)\alpha' \\ &= ((x, y)p(x, y)\alpha^{-1})(x, y)p(x, y)\alpha' \\ &= ((x, y)p\alpha)^{-1}(x, y)p\alpha'. \end{aligned} \tag{3.44}$$

Therefore, using Theorem 3.7, it is easy to see that α, α' are cohomologous if and only if $(S, \sigma, p\alpha, \mathbf{A})$ and $(S, \sigma, p\alpha', \mathbf{A})$ are equivalent. Hence, we have a well-defined injective map

$$\xi : [\alpha] \longrightarrow [S, \sigma, p\alpha, \mathbf{A}] : \mathbf{H}^2(S^l, \check{\mathbf{Z}}(\mathbf{A})^0\mathbf{F}) \longrightarrow \varepsilon(S, \mathbf{A}), \tag{3.45}$$

where $[S, \sigma, p\alpha, \mathbf{A}]$ denotes the equivalence class of $(S, \sigma, p\alpha, \mathbf{A})$. Let $(S, \sigma, q, \mathbf{A})$ be an extension of S by \mathbf{A} with abstract kernel Ψ . Then by (2.23), Lemma 1.6, and Proposition 2.5, we prove $(x, y)q(x, y)p^{-1} \in \mathbf{Z}(\mathbf{A})_{(xy)*xy} = \overline{\mathbf{Z}(\mathbf{A})}_{xy}$. Put $(x, y)\alpha = (x, y)q(x, y)p^{-1}$. Then $\alpha : S \times S \rightarrow \overline{\mathbf{Z}(\mathbf{A})}$, $(x, y)\alpha \in \overline{\mathbf{Z}(\mathbf{A})}_{xy}$ is a 2-cochain. α is indeed a 2-cocycle. So $[\alpha] \in \mathbf{H}^2(S^I, \check{\mathbf{Z}}(\mathbf{A})^0\mathbf{F})$ and $[\alpha]\xi = [S, \sigma, p\alpha, \mathbf{A}] = [S, \sigma, q, \mathbf{A}]$. Since every extension of S by \mathbf{A} with abstract kernel Ψ is equivalent to an extension of the form $(S, \sigma, q, \mathbf{A})$ by Theorem 2.18, it follows that ξ is surjective. The proof of the theorem is complete. \square

Theorems 3.6 and 3.8 generalize the corresponding results for inverse semigroups due to Lausch [4].

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