

ON MULTIPLE HARDY-HILBERT INTEGRAL INEQUALITIES WITH SOME PARAMETERS

HONG YONG

Received 19 April 2006; Revised 30 May 2006; Accepted 5 June 2006

By introducing some parameters and norm $\|x\|_\alpha$ ($x \in \mathbb{R}^n$), we give multiple Hardy-Hilbert integral inequalities, and prove that their constant factors are the best possible when parameters satisfy appropriate conditions.

Copyright © 2006 Hong Yong. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

1. Introduction

If $p > 1$, $1/p + 1/q = 1$, $f \geq 0$, $g \geq 0$, $0 < \int_0^\infty f^p(x)dx < +\infty$, $0 < \int_0^\infty g^q(x)dx < +\infty$, then we have the well-known Hardy-Hilbert inequality (see [4]):

$$\iint_0^{+\infty} \frac{f(x)g(x)}{x+y} dx dy < \frac{\pi}{\sin(\pi/p)} \left(\int_0^{+\infty} f^p(x)dx \right)^{1/p} \left(\int_0^{+\infty} g^q(x)dx \right)^{1/q}, \quad (1.1)$$

where the constant factor $\pi/\sin(\pi/p)$ is the best possible. Its equivalent form is

$$\int_0^{+\infty} \left(\int_0^{+\infty} \frac{f(x)}{x+y} dx \right)^p dy < \left[\frac{\pi}{\sin(\pi/p)} \right]^p \int_0^{+\infty} f^p(x)dx, \quad (1.2)$$

where the constant factor $[\pi/\sin(\pi/p)]^p$ is also the best possible.

Hardy-Hilbert inequalities are important in analysis and in their applications (see [7]). In recent years, many results (see [1, 3, 8–10]) have been obtained in the research of Hardy-Hilbert inequality. At present, because of the requirement of higher-dimensional harmonic analysis and higher-dimensional operator theory, multiple Hardy-Hilbert integral inequalities are researched (see [5, 6, 11]). Yang [11] obtains the following: if $\alpha \in \mathbb{R}$, $n \geq 2$, $p_i > 1$ ($i = 1, 2, \dots, n$), $\sum_{i=1}^n (1/p_i) = 1$, $\lambda > n - \min_{1 \leq i \leq n} \{p_i\}$, $f_i \geq 0$, and

2 Multiple Hardy-Hilbert integral inequalities

$0 < \int_{\alpha}^{+\infty} (t - \alpha)^{n-1-\lambda} f_i^{p_i}(t) dt < +\infty$, ($i = 1, 2, \dots, n$), then

$$\begin{aligned} & \int_{\alpha}^{+\infty} \cdots \int_{\alpha}^{+\infty} \frac{1}{(\sum_{i=1}^n x_i - n\alpha)^{\lambda}} \prod_{i=1}^n f_i(x_i) dx_1 \dots dx_n \\ & < \frac{1}{\Gamma(\lambda)} \prod_{i=1}^n \Gamma\left(1 - \frac{n-\lambda}{p_i}\right) \left[\int_{\alpha}^{+\infty} (t - \alpha)^{n-1-\lambda} f_i^{p_i}(t) dt \right]^{1/p_i}, \end{aligned} \quad (1.3)$$

where the constant factor $(1/\Gamma(\lambda)) \prod_{i=1}^n \Gamma(1 - (n-\lambda)/p_i)$ is the best possible.

In this paper, by introducing some parameters and norm $\|x\|_{\alpha}$ ($x \in \mathbb{R}^n$), we give multiple Hardy-Hilbert integral inequalities, and discuss the problem of the best constant factor. For this reason, we introduce the notation

$$\begin{aligned} R_+^n &= \{x = (x_1, \dots, x_n) : x_1, \dots, x_n > 0\}, \\ \|x\|_{\alpha} &= (x_1^{\alpha} + \cdots + x_n^{\alpha})^{1/\alpha}, \quad (\alpha > 0), \end{aligned} \quad (1.4)$$

and we agree on $\|x\|_{\alpha} < c$ representing $\{x \in \mathbb{R}_+^n : \|x\|_{\alpha} < c\}$.

2. Some lemmas

LEMMA 2.1 (see [2]). *If $p_i > 0$, $a_i > 0$, $\alpha_i > 0$, ($i = 1, 2, \dots, n$), $\Psi(u)$ is a measurable function, then*

$$\begin{aligned} & \int \cdots \int_{x_1, \dots, x_n > 0; (x_1/a_1)^{\alpha_1} + \cdots + (x_n/a_n)^{\alpha_n} \leq 1} \Psi\left(\left(\frac{x_1}{a_1}\right)^{\alpha_1} + \cdots + \left(\frac{x_n}{a_n}\right)^{\alpha_n}\right) \\ & \quad \times x_1^{p_1-1} \cdots x_n^{p_n-1} dx_1 \cdots dx_n \\ & = \frac{a_1^{p_1} \cdots a_n^{p_n} \Gamma(p_1/\alpha_1) \cdots \Gamma(p_n/\alpha_n)}{\alpha_1 \cdots \alpha_n \Gamma(p_1/\alpha_1 + \cdots + p_n/\alpha_n)} \int_0^1 \Psi(u) u^{p_1/\alpha_1 + \cdots + p_n/\alpha_n - 1} du, \end{aligned} \quad (2.1)$$

where the $\Gamma(\cdot)$ is Γ -function.

LEMMA 2.2. *If $n \in \mathbb{Z}_+$, $\alpha > 0$, $\beta > 0$, $\lambda > 0$, $m \in \mathbb{R}$, $0 < n - m < \beta\lambda$, and setting weight function $\omega_{\alpha, \beta, \lambda}(m, n, y)$ as*

$$\omega_{\alpha, \beta, \lambda}(m, n, y) = \int_{R_+^n} \frac{1}{(\|x\|_{\alpha}^{\beta} + \|y\|_{\alpha}^{\beta})^{\lambda}} \|x\|_{\alpha}^{-m} dx, \quad (2.2)$$

then

$$\omega_{\alpha,\beta,\lambda}(m,n,y) = \|y\|_{\alpha}^{n-\beta\lambda-m} \frac{\Gamma^n(1/\alpha)}{\beta\alpha^{n-1}\Gamma(n/\alpha)} B\left(\frac{n-m}{\beta}, \lambda - \frac{n-m}{\beta}\right), \quad (2.3)$$

where the $B(\cdot, \cdot)$ is β -function.

Proof. By Lemma 2.1, we have

$$\begin{aligned} \omega_{\alpha,\beta,\lambda}(m,n,y) &= \int_{\mathbb{R}_+^n} \frac{1}{(\|x\|_{\alpha}^{\beta} + \|y\|_{\alpha}^{\beta})^{\lambda}} \|x\|_{\alpha}^{-m} dy \\ &= \lim_{r \rightarrow +\infty} \int \cdots \int_{x_1, \dots, x_n > 0; x_1^{\alpha} + \cdots + x_n^{\alpha} < r^{\alpha}} \\ &\quad \times \frac{[r((x_1/r)^{\alpha} + \cdots + (x_n/r)^{\alpha})^{1/\alpha}]^{-m}}{[r^{\beta}((x_1/r)^{\alpha} + \cdots + (x_n/r)^{\alpha})^{\beta/\alpha} + \|y\|_{\alpha}^{\beta}]^{\lambda}} x_1^{1-1} \cdots x_n^{1-1} dx_1 \cdots dx_n \\ &= \lim_{r \rightarrow +\infty} \frac{r^n \Gamma^n(1/\alpha)}{\alpha^n \Gamma(n/\alpha)} \int_0^1 \frac{(ru^{1/\alpha})^{-m}}{(\|y\|_{\alpha}^{\beta} + r^{\beta} u^{\beta/\alpha})^{\lambda}} u^{n/\alpha-1} du \\ &= \frac{\Gamma^n(1/\alpha)}{\alpha^{n-1} \Gamma(n/\alpha)} \lim_{r \rightarrow +\infty} \int_0^r \frac{1}{(\|y\|_{\alpha}^{\beta} + t^{\beta})^{\lambda}} t^{n-m-1} dt \\ &= \frac{\Gamma^n(1/\alpha)}{\alpha^{n-1} \Gamma(n/\alpha)} \int_0^{+\infty} \frac{1}{(\|y\|_{\alpha}^{\beta} + t^{\beta})^{\lambda}} t^{n-m-1} dt \\ &= \|y\|_{\alpha}^{n-\beta\lambda-m} \frac{\Gamma^n(1/\alpha)}{\beta\alpha^{n-1} \Gamma(n/\alpha)} \int_0^1 \frac{1}{(1+u)^{\lambda}} u^{(n-m)/\beta-1} du \\ &= \|y\|_{\alpha}^{n-\beta\lambda-m} \frac{\Gamma^n(1/\alpha)}{\beta\alpha^{n-1} \Gamma(n/\alpha)} B\left(\frac{n-m}{\beta}, \lambda - \frac{n-m}{\beta}\right). \end{aligned} \quad (2.4)$$

Hence (2.3) is valid. □

3. Main results

THEOREM 3.1. *If $p > 1$, $1/p + 1/q = 1$, $n \in \mathbb{Z}_+$, $\alpha > 0$, $\beta > 0$, $\lambda > 0$, $a \in \mathbb{R}$, $b \in \mathbb{R}$, $0 < n - ap < \beta\lambda$, $0 < n - bq < \beta\lambda$, $f \geq 0$, $g \geq 0$, and*

$$0 < \int_{\mathbb{R}_+^n} \|x\|_{\alpha}^{(n-\beta\lambda)+p(b-a)} f^p(x) dx < +\infty, \quad (3.1)$$

$$0 < \int_{\mathbb{R}_+^n} \|y\|_{\alpha}^{(n-\beta\lambda)+q(a-b)} g^q(y) dy < +\infty, \quad (3.2)$$

4 Multiple Hardy-Hilbert integral inequalities

then

$$\begin{aligned} & \iint_{R_+^n} \frac{f(x)g(y)}{(\|x\|_\alpha^\beta + \|y\|_\alpha^\beta)^\lambda} dx dy \\ & < C_{\alpha,\beta,\lambda}(a,b,p,q) \times \left(\int_{R_+^n} \|x\|_\alpha^{(n-\beta\lambda)+p(b-a)} f^p(x) dx \right)^{1/p} \left(\int_{R_+^n} \|y\|_\alpha^{(n-\beta\lambda)+q(a-b)} g^q(y) dy \right)^{1/q}, \end{aligned} \quad (3.3)$$

$$\begin{aligned} & \int_{R_+^n} \|y\|_\alpha^{((n-\beta\lambda)+q(a-b))/(1-q)} \left[\int_{R_+^n} \frac{f(x)}{(\|x\|_\alpha^\beta + \|y\|_\alpha^\beta)^\lambda} dx \right]^p dy \\ & < C_{\alpha,\beta,\lambda}^p(a,b,p,q) \times \int_{R_+^n} \|x\|_\alpha^{(n-\beta\lambda)+p(b-a)} f^p(x) dx, \end{aligned} \quad (3.4)$$

where $C_{\alpha,\beta,\lambda}(a,b,p,q) = (\Gamma^n(1/\alpha)/\beta\alpha^{n-1}\Gamma(n/\alpha))B^{1/p}((n-ap)/\beta,\lambda - (n-ap)/\beta)B^{1/q}((n-bq)/\beta,\lambda - (n-bq)/\beta)$.

Proof. By Hölder's inequality, we have

$$\begin{aligned} G &:= \iint_{R_+^n} \frac{f(x)g(y)}{(\|x\|_\alpha^\beta + \|y\|_\alpha^\beta)^\lambda} dx dy \\ &= \iint_{R_+^n} \left(\frac{f(x)}{(\|x\|_\alpha^\beta + \|y\|_\alpha^\beta)^{\lambda/p}} \frac{\|x\|_\alpha^{b/p}}{\|y\|_\alpha^{a/p}} \right) \left(\frac{g(y)}{(\|x\|_\alpha^\beta + \|y\|_\alpha^\beta)^{\lambda/q}} \frac{\|y\|_\alpha^{a/q}}{\|x\|_\alpha^{b/q}} \right) dx dy \\ &\leq \left(\iint_{R_+^n} \frac{f^p(x)}{(\|x\|_\alpha^\beta + \|y\|_\alpha^\beta)^\lambda} \frac{\|x\|_\alpha^{bp}}{\|y\|_\alpha^{ap}} dx dy \right)^{1/p} \times \left(\iint_{R_+^n} \frac{g^q(y)}{(\|x\|_\alpha^\beta + \|y\|_\alpha^\beta)^\lambda} \frac{\|y\|_\alpha^{aq}}{\|x\|_\alpha^{bq}} dx dy \right)^{1/q}, \end{aligned} \quad (3.5)$$

according to the condition of taking equality in Hölder's inequality, if this inequality takes the form of an equality, then there exist constants C_1 and C_2 , such that they are not all zero, and

$$\frac{C_1 f^p(x)}{(\|x\|_\alpha^\beta + \|y\|_\alpha^\beta)^\lambda} \frac{\|x\|_\alpha^{bp}}{\|y\|_\alpha^{ap}} = \frac{C_2 g^q(y)}{(\|x\|_\alpha^\beta + \|y\|_\alpha^\beta)^\lambda} \frac{\|y\|_\alpha^{aq}}{\|x\|_\alpha^{bq}}, \quad \text{a.e. } (x,y) \in R_+^n \times R_+^n. \quad (3.6)$$

Without losing generality, we suppose that $C_1 \neq 0$, we may get

$$\|x\|_\alpha^{b(p+q)} f^p(x) = \frac{C_2}{C_1} \|y\|_\alpha^{a(p+q)} g^q(y), \quad \text{a.e. } (x,y) \in R_+^n \times R_+^n, \quad (3.7)$$

hence, we obtain

$$\|x\|_\alpha^{b(p+q)} f^p(x) = C(\text{constant}), \quad \text{a.e. } x \in R_+^n, \quad (3.8)$$

hence, we have

$$\begin{aligned} \int_{R_+^n} \|x\|_\alpha^{(n-\beta\lambda)+p(b-a)} f^p(x) dx &= \int_{R_+^n} \|x\|_\alpha^{(n-\beta\lambda)-bq-ap+b(p+q)} f^p(x) dx \\ &= C \int_{R_+^n} \|x\|_\alpha^{(n-\beta\lambda)-bq-ap} dx = \infty, \end{aligned} \quad (3.9)$$

which contradicts (3.1). Hence, and by Lemma 2.2, we obtain

$$\begin{aligned} G &< \left[\int_{R_+^n} \left(\int_{R_+^n} \frac{1}{(\|x\|_\alpha^\beta + \|y\|_\alpha^\beta)^\lambda} \frac{1}{\|y\|_\alpha^{ap}} dy \right) \|x\|_\alpha^{bp} f^p(x) dx \right]^{1/p} \\ &\quad \times \left[\int_{R_+^n} \left(\int_{R_+^n} \frac{1}{(\|x\|_\alpha^\beta + \|y\|_\alpha^\beta)^\lambda} \frac{1}{\|x\|_\alpha^{bq}} dx \right) \|y\|_\alpha^{aq} g^q(y) dy \right]^{1/q} \\ &= \left(\int_{R_+^n} \omega_{\alpha,\beta,\lambda}(ap, n, x) \|x\|_\alpha^{bp} f^p(x) dx \right)^{1/p} \left(\int_{R_+^n} \omega_{\alpha,\beta,\lambda}(bq, n, y) \|y\|_\alpha^{aq} g^q(y) dy \right)^{1/q} \\ &= \left[\frac{\Gamma^n(1/\alpha)}{\beta\alpha^{n-1}\Gamma(n/\alpha)} B\left(\frac{n-ap}{\beta}, \lambda - \frac{n-ap}{\beta}\right) \int_{R_+^n} \|x\|_\alpha^{(n-\beta\lambda)+p(b-a)} f^p(x) dx \right]^{1/p} \\ &\quad \times \left[\frac{\Gamma^n(1/\alpha)}{\beta\alpha^{n-1}\Gamma(n/\alpha)} B\left(\frac{n-bq}{\beta}, \lambda - \frac{n-bq}{\beta}\right) \int_{R_+^n} \|y\|_\alpha^{(n-\beta\lambda)+q(a-b)} g^q(y) dy \right]^{1/q} \\ &= C_{\alpha,\beta,\lambda}(a, b, p, q) \left[\int_{R_+^n} \|x\|_\alpha^{(n-\beta\lambda)+p(b-a)} f^p(x) dx \right]^{1/p} \times \left[\int_{R_+^n} \|y\|_\alpha^{(n-\beta\lambda)+q(a-b)} g^q(y) dy \right]^{1/q}. \end{aligned} \quad (3.10)$$

Hence, (3.3) is valid.

Let $k = ((n - \beta\lambda) + q(a - b))/(1 - q)$, for $0 < h < l < +\infty$, setting

$$g_{h,l}(y) = \begin{cases} \|y\|_\alpha^k \left(\int_{R_+^n} \frac{f(x)}{(\|x\|_\alpha^\beta + \|y\|_\alpha^\beta)^\lambda} dx \right)^{p/q}, & h < \|y\|_\alpha < l, \\ 0, & 0 < \|y\|_\alpha \leq h \text{ or } \|y\|_\alpha \geq l, \end{cases} \quad (3.11)$$

$$\tilde{g}(y) = \|y\|_\alpha^k \left(\int_{R_+^n} \frac{f(x)}{(\|x\|_\alpha^\beta + \|y\|_\alpha^\beta)^\lambda} dx \right)^{p/q}, \quad y \in R_+^n,$$

by (3.1), for sufficiently small $h > 0$ and sufficiently large $l > 0$, we have

$$0 < \int_{h < \|y\|_\alpha < l} \|y\|_\alpha^{(n-\beta\lambda)+q(a-b)} g_{h,l}^q(y) dy < +\infty. \quad (3.12)$$

6 Multiple Hardy-Hilbert integral inequalities

Hence, by (3.3), we have

$$\begin{aligned}
& \int_{h < \|y\|_\alpha < l} \|y\|_\alpha^{(n-\beta\lambda)+q(a-b)} \tilde{g}^q(y) dy \\
&= \int_{h < \|y\|_\alpha < l} \|y\|_\alpha^{k(1-q)} \tilde{g}^q(y) dy = \int_{h < \|y\|_\alpha < l} \|y\|_\alpha^k \left(\int_{R_+^n} \frac{f(x)}{(\|x\|_\alpha^\beta + \|y\|_\alpha^\beta)^\lambda} dx \right)^p dy \\
&= \int_{h < \|y\|_\alpha < l} \|y\|_\alpha^k \left(\int_{R_+^n} \frac{f(x)}{(\|x\|_\alpha^\beta + \|y\|_\alpha^\beta)^\lambda} dx \right)^{p/q} \left(\int_{R_+^n} \frac{f(x)}{(\|x\|_\alpha^\beta + \|y\|_\alpha^\beta)^\lambda} dx \right) dy \\
&= \iint_{R_+^n} \frac{f(x) g_{h,l}(y)}{(\|x\|_\alpha^\beta + \|y\|_\alpha^\beta)^\lambda} dx dy < C_{\alpha,\beta,\lambda}(a,b,p,q) \left(\int_{R_+^n} \|x\|_\alpha^{(n-\beta\lambda)+p(b-a)} f^p(x) dx \right)^{1/p} \\
&\quad \times \left(\int_{R_+^n} \|y\|_\alpha^{(n-\beta\lambda)+q(a-b)} g_{h,l}^q(y) dy \right)^{1/q} = C_{\alpha,\beta,\lambda}(a,b,p,q) \left(\int_{R_+^n} \|x\|_\alpha^{(n-\beta\lambda)+p(b-a)} f^p(x) dx \right)^{1/p} \\
&\quad \times \left(\int_{h < \|y\|_\alpha < l} \|y\|_\alpha^{(n-\beta\lambda)+q(a-b)} \tilde{g}^q(y) dy \right)^{1/q}, \tag{3.13}
\end{aligned}$$

it follows that

$$\int_{h < \|y\|_\alpha < l} \|y\|_\alpha^{(n-\beta\lambda)+q(a-b)} \tilde{g}^q(y) dy < C_{\alpha,\beta,\lambda}^p(a,b,p,q) \int_{R_+^n} \|x\|_\alpha^{(n-\beta\lambda)+p(b-a)} f^p(x) dx. \tag{3.14}$$

For $h \rightarrow 0^+$, $l \rightarrow +\infty$, we obtain

$$\begin{aligned}
0 &< \int_{R_+^n} \|y\|_\alpha^{(n-\beta\lambda)+q(a-b)} \tilde{g}^q(y) dy \\
&\leq C_{\alpha,\beta,\lambda}^p(a,b,p,q) \int_{R_+^n} \|x\|_\alpha^{(n-\beta\lambda)+p(b-a)} f^p(x) dx < +\infty, \tag{3.15}
\end{aligned}$$

hence, by (3.3), we obtain

$$\begin{aligned}
& \int_{R_+^n} \|y\|_\alpha^{((n-\beta\lambda)+q(a-b))/(1-q)} \left(\int_{R_+^n} \frac{f(x)}{(\|x\|_\alpha^\beta + \|y\|_\alpha^\beta)^\lambda} dx \right)^p dy \\
&= \iint_{R_+^n} \frac{f(x) \tilde{g}(y)}{(\|x\|_\alpha^\beta + \|y\|_\alpha^\beta)^\lambda} dx dy < C_{\alpha,\beta,\lambda}(a,b,p,q) \left(\int_{R_+^n} \|x\|_\alpha^{(n-\beta\lambda)+p(b-a)} f^p(x) dx \right)^{1/p} \\
&\quad \times \left(\int_{R_+^n} \|y\|_\alpha^{(n-\beta\lambda)+q(a-b)} \tilde{g}^q(y) dy \right)^{1/q} = C_{\alpha,\beta,\lambda}(a,b,p,q) \left(\int_{R_+^n} \|x\|_\alpha^{(n-\beta\lambda)+p(b-a)} f^p(x) dx \right)^{1/p} \\
&\quad \times \left[\int_{R_+^n} \|y\|_\alpha^{((n-\beta\lambda)+q(a-b))/(1-q)} \left(\int_{R_+^n} \frac{f(x)}{(\|x\|_\alpha^\beta + \|y\|_\alpha^\beta)^\lambda} dx \right)^p dy \right]^{1/q}. \tag{3.16}
\end{aligned}$$

Hence, we can obtain (3.4). \square

Remark 3.2. If f and g do not satisfy (3.1) and (3.2), by the proof of Theorem 3.1, we can obtain

$$\begin{aligned} & \iint_{\mathbb{R}_+^n} \frac{f(x)g(y)}{(\|x\|_\alpha^\beta + \|y\|_\alpha^\beta)^\lambda} dx dy \\ & \leq C_{\alpha,\beta,\lambda}(a,b,p,q) \times \left(\int_{\mathbb{R}_+^n} \|x\|_\alpha^{(n-\beta\lambda)+p(b-a)} f^p(x) dx \right)^{1/p} \left(\int_{\mathbb{R}_+^n} \|y\|_\alpha^{(n-\beta\lambda)+q(a-b)} g^q(y) dy \right)^{1/q}, \end{aligned} \quad (3.17)$$

$$\begin{aligned} & \int_{\mathbb{R}_+^n} \|y\|_\alpha^{((n-\beta\lambda)+q(a-b))/(1-q)} \left[\int_{\mathbb{R}_+^n} \frac{f(x)}{(\|x\|_\alpha^\beta + \|y\|_\alpha^\beta)^\lambda} dx \right]^p dy \\ & \leq C_{\alpha,\beta,\lambda}^p(a,b,p,q) \times \int_{\mathbb{R}_+^n} \|x\|_\alpha^{(n-\beta\lambda)+p(b-a)} f^p(x) dx. \end{aligned} \quad (3.18)$$

Remark 3.3. By (3.4), we can also obtain (3.3), hence (3.4) and (3.3) are equivalent.

THEOREM 3.4. If $p > 1$, $1/p + 1/q = 1$, $n \in \mathbb{Z}_+$, $\alpha > 0$, $\beta > 0$, $\lambda > 0$, $a \in \mathbb{R}$, $b \in \mathbb{R}$, $0 < n - ap < \beta\lambda$, $ap + bq = 2n - \beta\lambda$, $f \geq 0$, $g \geq 0$, and

$$\begin{aligned} 0 & < \int_{\mathbb{R}_+^n} \|x\|_\alpha^{b(p+q)-n} f^p(x) dx < +\infty, \\ 0 & < \int_{\mathbb{R}_+^n} \|y\|_\alpha^{a(p+q)-n} g^q(y) dy < +\infty, \end{aligned} \quad (3.19)$$

then

$$\begin{aligned} & \iint_{\mathbb{R}_+^n} \frac{f(x)g(y)}{(\|x\|_\alpha^\beta + \|y\|_\alpha^\beta)^\lambda} dx dy \\ & < \frac{\Gamma^n(1/\alpha)}{\beta\alpha^{n-1}\Gamma(n/\alpha)} B\left(\frac{n-ap}{\beta}, \lambda - \frac{n-ap}{\beta}\right) \\ & \quad \times \left(\int_{\mathbb{R}_+^n} \|x\|_\alpha^{b(p+q)-n} f^p(x) dx \right)^{1/p} \left(\int_{\mathbb{R}_+^n} \|y\|_\alpha^{a(p+q)-n} g^q(y) dy \right)^{1/q}, \end{aligned} \quad (3.20)$$

$$\begin{aligned} & \int_{\mathbb{R}_+^n} \|y\|_\alpha^{(a(p+q)-n)/(1-q)} \left[\int_{\mathbb{R}_+^n} \frac{f(x)}{(\|x\|_\alpha^\beta + \|y\|_\alpha^\beta)^\lambda} dx \right]^p dy \\ & < \left[\frac{\Gamma^n(1/\alpha)}{\beta\alpha^{n-1}\Gamma(n/\alpha)} B\left(\frac{n-ap}{\beta}, \lambda - \frac{n-ap}{\beta}\right) \right]^p \int_{\mathbb{R}_+^n} \|x\|_\alpha^{b(p+q)-n} f^p(x) dx, \end{aligned} \quad (3.21)$$

where the constant factors $(\Gamma^n(1/\alpha)/\beta\alpha^{n-1}\Gamma(n/\alpha))B((n-ap)/\beta, \lambda - (n-ap)/\beta)$ and $[(\Gamma^n(1/\alpha)/\beta\alpha^{n-1}\Gamma(n/\alpha))B((n-ap)/\beta, \lambda - (n-ap)/\beta)]^p$ are all the best possible.

Proof. Since $ap + bq = 2n - \beta\lambda$, we have

$$n - bq = n - (2n - \beta\lambda - ap) = \beta\lambda - (n - ap), \quad (3.22)$$

8 Multiple Hardy-Hilbert integral inequalities

hence, by $0 < n - ap < \beta\lambda$, we obtain $0 < n - bq < \beta\lambda$, and

$$(n - \beta\lambda) + p(b - a) = b(p + q) - n, \quad (n - \beta\lambda) + q(a - b) = a(p + q) - n, \\ \frac{n - ap}{\beta} = \lambda - \frac{n - bq}{\beta}, \quad \lambda - \frac{n - ap}{\beta} = \frac{n - bq}{\beta}. \quad (3.23)$$

By Theorem 3.1, (3.20) and (3.21) are valid.

If the constant factor $K_1 := (\Gamma^n(1/\alpha)/\beta\alpha^{n-1}\Gamma(n/\alpha))B((n - ap)/\beta, \lambda - (n - ap)/\beta)$ in (3.20) is not the best possible, then there exists a positive constant $K < K_1$, such that (3.20) is still valid when we replace K_1 by K .

In particular, for $0 < \varepsilon < q(n - ap)$, we take

$$f_\varepsilon(x) = \|x\|_\alpha^{-bq - \varepsilon/p}, \quad g_\varepsilon(y) = \|y\|_\alpha^{-ap - \varepsilon/q}, \quad (3.24)$$

by (3.17) and the properties of limit, when $\delta > 0$ is sufficiently small, we have

$$\int_{\|x\|_\alpha > \delta} \int_{R_+^n} \frac{f_\varepsilon(x)g_\varepsilon(y)}{(\|x\|_\alpha^\beta + \|y\|_\alpha^\beta)^\lambda} dx dy \\ \leq K \left(\int_{\|x\|_\alpha > \delta} \|x\|_\alpha^{b(p+q) - n} f_\varepsilon^p(x) dx \right)^{1/p} \left(\int_{\|y\|_\alpha > \delta} \|y\|_\alpha^{a(p+q) - n} g_\varepsilon^q(y) dy \right)^{1/q} \\ = K \left(\int_{\|x\|_\alpha > \delta} \|x\|_\alpha^{-n - \varepsilon} \right)^{1/p} \left(\int_{\|y\|_\alpha > \delta} \|y\|_\alpha^{-n - \varepsilon} dy \right)^{1/q} = K \int_{\|x\|_\alpha > \delta} \|x\|_\alpha^{-n - \varepsilon} dx. \quad (3.25)$$

On the other hand, by Lemma 2.2, we have

$$\int_{\|x\|_\alpha > \delta} \int_{R_+^n} \frac{f_\varepsilon(x)g_\varepsilon(y)}{(\|x\|_\alpha^\beta + \|y\|_\alpha^\beta)^\lambda} dx dy \\ = \int_{\|x\|_\alpha > \delta} \|x\|_\alpha^{-bq - \varepsilon/p} \int_{R_+^n} \frac{1}{(\|x\|_\alpha^\beta + \|y\|_\alpha^\beta)^\lambda} \|y\|_\alpha^{-ap - \varepsilon/q} dy dx \\ = \int_{\|x\|_\alpha > \delta} \|x\|_\alpha^{-bq - \varepsilon/p} \tilde{\omega}_{\alpha, \beta, \lambda} \left(ap + \frac{\varepsilon}{q}, n, x \right) dx \\ = \frac{\Gamma^n(1/\alpha)}{\beta\alpha^{n-1}\Gamma(n/\alpha)} B \left(\frac{1}{\beta} \left(n - ap - \frac{\varepsilon}{q} \right), \lambda - \frac{1}{\beta} \left(n - ap - \frac{\varepsilon}{q} \right) \right) \int_{\|x\|_\alpha > \delta} \|x\|_\alpha^{-n - \varepsilon} dx. \quad (3.26)$$

Hence, we obtain

$$\frac{\Gamma^n(1/\alpha)}{\beta\alpha^{n-1}\Gamma(n/\alpha)} B \left(\frac{1}{\beta} \left(n - ap - \frac{\varepsilon}{q} \right), \lambda - \frac{1}{\beta} \left(n - ap - \frac{\varepsilon}{q} \right) \right) \leq K, \quad (3.27)$$

for $\varepsilon \rightarrow 0^+$, we have

$$K_1 = \frac{\Gamma^n(1/\alpha)}{\beta\alpha^{n-1}\Gamma(n/\alpha)} B \left(\frac{n - ap}{\beta}, \lambda - \frac{n - ap}{\beta} \right) \leq K, \quad (3.28)$$

which contradicts the fact that $K < K_1$. Hence the constant factor in (3.20) is the best possible.

Since (3.21) and (3.20) are equivalent, the constant factor in (3.21) is also the best possible. \square

4. Some corollaries

COROLLARY 4.1. *If $p > 1$, $1/p + 1/q = 1$, $n \in \mathbb{Z}_+$, $\alpha > 0$, $\beta > 0$, $\lambda > 0$, $f \geq 0$, $g \geq 0$, and*

$$\begin{aligned} 0 &< \int_{\mathbb{R}_+^n} \|x\|_\alpha^{(n-\beta\lambda)(p-1)} f^p(x) dx < +\infty, \\ 0 &< \int_{\mathbb{R}_+^n} \|y\|_\alpha^{(n-\beta\lambda)(q-1)} g^q(y) dy < +\infty, \end{aligned} \tag{4.1}$$

then

$$\begin{aligned} &\iint_{\mathbb{R}_+^n} \frac{f(x)g(y)}{(\|x\|_\alpha^\beta + \|y\|_\alpha^\beta)^\lambda} dx dy \\ &< \frac{\Gamma^n(1/\alpha)}{\beta\alpha^{n-1}\Gamma(n/\alpha)} B\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right) \left(\int_{\mathbb{R}_+^n} \|x\|_\alpha^{(n-\beta\lambda)(p-1)} f^p(x) dx\right)^{1/p} \left(\int_{\mathbb{R}_+^n} \|y\|_\alpha^{(n-\beta\lambda)(q-1)} g^q(y) dy\right)^{1/q}, \\ &\int_{\mathbb{R}_+^n} \|y\|_\alpha^{\beta\lambda-n} \left[\int_{\mathbb{R}_+^n} \frac{f(x)}{(\|x\|_\alpha^\beta + \|y\|_\alpha^\beta)^\lambda} dx\right]^p dy \\ &< \left[\frac{\Gamma^n(1/\alpha)}{\beta\alpha^{n-1}\Gamma(n/\alpha)} B\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right)\right]^p \int_{\mathbb{R}_+^n} \|x\|_\alpha^{(n-\beta\lambda)(p-1)} f^p(x) dx, \end{aligned} \tag{4.2}$$

where the constant factors in (4.2) are all the best possible.

Proof. If we take $a = n/p - \beta\lambda/p^2$, $b = n/q - \beta\lambda/q^2$ in Theorem 3.4, (4.2) can be obtained. \square

Remark 4.2. If we take $n = \lambda = 1$ in (4.2), we can obtain the results of [10]:

$$\begin{aligned} &\iint_0^{+\infty} \frac{f(x)g(y)}{x^\beta + y^\beta} dx dy \\ &< \frac{\pi}{\beta \sin(\pi/p)} \left(\int_0^{+\infty} x^{(p-1)(1-\beta)} f^p(x) dx\right)^{1/p} \left(\int_0^{+\infty} y^{(q-1)(1-\beta)} g^q(y) dy\right)^{1/q}, \tag{4.3} \\ &\int_0^{+\infty} y^{\beta-1} \left(\int_0^{+\infty} \frac{f(x)}{x^\beta + y^\beta} dx\right)^p dy < \left(\frac{\pi}{\beta \sin(\pi/p)}\right)^p \int_0^{+\infty} x^{(p-1)(1-\beta)} f^p(x) dx, \end{aligned}$$

where the constant factors in (4.3) are all the best possible.

10 Multiple Hardy-Hilbert integral inequalities

If we take $n = \beta = 1$ in (4.2), we can obtain

$$\begin{aligned} & \iint_0^{+\infty} \frac{f(x)g(y)}{(x+y)^\lambda} dx dy \\ & < B\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right) \left(\int_0^{+\infty} x^{(1-\lambda)(p-1)} f^p(x) dx\right)^{1/p} \left(\int_0^{+\infty} y^{(1-\lambda)(q-1)} g^q(y) dy\right)^{1/q}, \quad (4.4) \\ & \int_0^{+\infty} y^{\lambda-1} \left(\int_0^{+\infty} \frac{f(x)}{(x+y)^\lambda} dx\right)^p dy < B^p\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right) \int_0^{+\infty} x^{(1-\lambda)(p-1)} f^p(x) dx, \end{aligned}$$

where the constant factors in (4.4) are all the best possible.

COROLLARY 4.3. *If $p > 1$, $1/p + 1/q = 1$, $n \in \mathbb{Z}_+$, $\lambda > 0$, $np + \lambda - 2n > 0$, $nq + \lambda - 2n > 0$, $f \geq 0$, $g \geq 0$, and*

$$\begin{aligned} 0 & < \int_{\mathbb{R}_+^n} \|x\|_\alpha^{n-\lambda} f^p(x) dx < +\infty, \\ 0 & < \int_{\mathbb{R}_+^n} \|y\|_\alpha^{n-\lambda} g^q(y) dy < +\infty, \end{aligned} \quad (4.5)$$

then

$$\begin{aligned} & \iint_{\mathbb{R}_+^n} \frac{f(x)g(y)}{(\|x\|_\alpha + \|y\|_\alpha)^\lambda} dx dy \\ & < B\left(\frac{np + \lambda - 2n}{p}, \frac{nq + \lambda - 2n}{q}\right) \left(\int_{\mathbb{R}_+^n} \|x\|_\alpha^{n-\lambda} f^p(x) dx\right)^{1/p} \left(\int_{\mathbb{R}_+^n} \|y\|_\alpha^{n-\lambda} g^q(y) dy\right)^{1/q}, \\ & \int_{\mathbb{R}_+^n} \|y\|_\alpha^{(n-\lambda)/(1-q)} \left[\int_{\mathbb{R}_+^n} \frac{f(x)}{(\|x\|_\alpha + \|y\|_\alpha)^\lambda} dx\right]^p dy < B^p\left(\frac{np + \lambda - 2n}{p}, \frac{nq + \lambda - 2n}{q}\right) \int_{\mathbb{R}_+^n} \|x\|_\alpha^{n-\lambda} f^p(x) dx, \end{aligned} \quad (4.6)$$

where the constant factors in (4.6) are all the best possible.

Proof. If we take $\beta = 1$, $a = b = (2n - \lambda)/pq$ in Theorem 3.4, (4.6) can be obtained. \square

Remark 4.4. If we take $n = 1$ in (4.6), we can obtain the results of [1]:

$$\begin{aligned} & \iint_0^{+\infty} \frac{f(x)g(y)}{(x+y)^\lambda} dx dy \\ & < B\left(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q}\right) \left(\int_0^{+\infty} x^{1-\lambda} f^p(x) dx\right)^{1/p} \left(\int_0^{+\infty} y^{1-\lambda} g^q(y) dy\right)^{1/q}, \\ & \int_0^{+\infty} y^{(1-\lambda)/(1-q)} \left(\int_0^{+\infty} \frac{f(x)}{(x+y)^\lambda} dx\right)^p dy < B^p\left(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q}\right) \int_0^{+\infty} x^{1-\lambda} f^p(x) dx, \end{aligned} \quad (4.7)$$

where the constant factors in (4.7) are all the best possible.

If we take other appropriate parameters, we can obtain many new inequalities.

References

- [1] Y. Bicheng, *On Hardy-Hilbert's integral inequality*, Journal of Mathematical Analysis and Applications **261** (2001), no. 1, 295–306.
- [2] G. M. Fichtingoloz, *A Course in Differential and Integral Calculus*, Renmin Jiaoyu, Beijing, 1959.
- [3] M. Gao, T. Li, and L. Debnath, *Some improvements on Hilbert's integral inequality*, Journal of Mathematical Analysis and Applications **229** (1999), no. 2, 682–689.
- [4] G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, Cambridge University Press, Cambridge, 1952.
- [5] Y. Hong, *All-sided generalization about Hardy-Hilbert integral inequalities*, Acta Mathematica Sinica (China) **44** (2001), no. 4, 619–626.
- [6] K. Jichang, *Applied Inequalities*, China Shandong Science and Technology Press, Jinan, 2004.
- [7] D. S. Mitrinović, J. E. Pečarić, and A. M. Fink, *Inequalities Involving Functions and Their Integrals and Derivatives*, Mathematics and Its Applications (East European Series), vol. 53, Kluwer Academic, Dordrecht, 1991.
- [8] B. G. Pachpatte, *On some new inequalities similar to Hilbert's inequality*, Journal of Mathematical Analysis and Applications **226** (1998), no. 1, 166–179.
- [9] B. Yang, *A general Hardy-Hilbert's integral inequality with a best constant*, Chinese Annals of Mathematics. Series A. **21** (2000), no. 4, 401–408.
- [10] ———, *An extension of Hardy-Hilbert's inequality*, Chinese Annals of Mathematics. Series A. **23** (2002), no. 2, 247–254.
- [11] ———, *A multiple Hardy-Hilbert integral inequality*, Chinese Annals of Mathematics. Series A. **24** (2003), no. 6, 743–750.

Hong Yong: Department of Mathematics, Guangdong University of Business Study,
Guangzhou 510320, China
E-mail address: hongyong59@sohu.com