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SQUARE-ROOT FAMILIES FOR THE SIMULTANEOUS APPROXIMATION OF POLYNOMIAL MULTIPLE ZEROS

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Abstract. One-parameter families of iterative methods for the simultaneous determination of multiple complex zeros of a polynomial are considered. Acceleration of convergence is performed by using Newton's and Halley's corrections for a multiple zero. It is shown that the convergence order of the constructed total-step methods is five and six, respectively. By applying the Gauss-Seidel approach, further improvements of these methods are obtained. The lower bounds of the R-order of convergence of the improved (single-step) methods are derived. Accelerated convergence of all proposed methods is attained with negligible number of additional operations, which provides a high computational efficiency of these methods. Convergence analysis and numerical results are given.

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1. Square-root families for multiple zeros

In this paper we give new high-order families of iterative methods for the simultaneous determination of all multiple (real or complex) zeros of a polynomial. The improved methods with accelerated convergence are constructed by using suitable corrections. They have a high computational efficiency since the accelerated convergence is attained with only negligible number of additional numerical operations. Convergence analysis of the proposed methods and numerical examples are given.

Let P be a monic polynomial of degree $n \geq 3$ with the zeros $\zeta_1, \ldots, \zeta_\nu$ of the multiplicities $\mu_1, \ldots, \mu_\nu$ $(\mu_1 + \cdots + \mu_\nu = n, 1 < \nu \leq n)$ and let $z_1, \ldots, z_\nu$ be their mutually distinct approximations. The determination of the order of multiplicity is not considered here; efficient multiplicity-finding algorithms may be found, for instance, in [2] and [3]. For the point $z = z_i$ $(i \in I_\nu := \{1, \ldots, \nu\})$ let us introduce the notations:

$$\Sigma_{\lambda,i} = \sum_{\substack{j=1 \\ j \neq i}}^{\nu} \frac{\mu_j}{(z_i - \zeta_j)^\lambda}, \quad S_{\lambda,i} = \sum_{\substack{j=1 \\ j \neq i}}^{\nu} \frac{\mu_j}{(z_i - z_j)^\lambda} \quad (\lambda = 1, 2),$$

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$$\delta_{1,i} = \frac{P'(z_i)}{P(z_i)}, \quad \delta_{2,i} = \frac{P'(z_i)^2 - P(z_i)P''(z_i)}{P(z_i)^2}$$

$$f_i^* = \mu_i(\alpha+1)\Sigma_{2,i} - \alpha(\alpha+1)\Sigma_{1,i}^2, \tag{1}$$

$$f_i = \mu_i(\alpha+1)S_{2,i} - \alpha(\alpha+1)S_{1,i}^2 \tag{2}$$

$$\varepsilon_i = z_i - \zeta_i, \quad \varepsilon = \max_{1\le i\le \nu} |\varepsilon_i|.$$

Starting from the identities

$$\delta_{1,i} = \frac{P'(z_i)}{P(z_i)} = \sum_{j=1}^{\nu} \frac{\mu_j}{z_i - \zeta_j} = \frac{\mu_i}{z_i - \zeta_i} + \Sigma_{1,i} \tag{3}$$

and

$$\begin{aligned} \delta_{2,i} &= \frac{P'(z_i)^2 - P(z_i)P''(z_i)}{P(z_i)^2} = -\left(\frac{P'(z)}{P(z)}\right)'\Bigg|_{z=z_i} = \sum_{j=1}^{\nu} \frac{\mu_j}{(z_i-\zeta_j)^2} \\ &= \frac{\mu_i}{(z_i-\zeta_i)^2} + \Sigma_{2,i}, \end{aligned} \tag{4}$$

we come to the identity

$$\mu_i(\alpha+1)\delta_{2,i} - \alpha\delta_{1,i}^2 - f_i^* = \left(\frac{\mu_i(\alpha+1)}{\varepsilon_i} - \alpha\delta_{1,i}\right)^2. \tag{5}$$

From the identity (5) we derive the following fixed-point relation

$$\zeta_i = z_i - \frac{\mu_i(\alpha+1)}{\alpha\delta_{1,i} \pm \left[\mu_i(\alpha+1)\delta_{2,i} - \alpha\delta_{1,i}^2 - f_i^*\right]^{1/2}} \tag{6}$$

for $i \in I_\nu$, assuming that two values of the square root have to be taken in (6).

Let us introduce some abbreviations in a similar way as in [7]:

1° The approximations $z_1^{(m)}, \dots, z_\nu^{(m)}$ of the zeros at the mth iterative step will be briefly denoted by $z_1, \dots, z_\nu$, and the new approximations $z_1^{(m+1)}, \dots, z_\nu^{(m+1)}$, obtained by some simultaneous iterative method, by $\hat{z}_1, \dots, \hat{z}_\nu$, respectively;

2°

$$N_i = N(z_i) = \frac{\mu_i}{\delta_{1,i}} = \mu_i \frac{P(z_i)}{P'(z_i)} \quad \text{(Schröder's correction)}, \tag{7}$$

$$\begin{aligned} H_i &= H(z_i) = \left[\frac{P'(z_i)}{2P(z_i)}\left(1 + \frac{1}{\mu_i}\right) - \frac{P''(z_i)}{2P'(z_i)}\right]^{-1} \\ &= \frac{2\mu_i\delta_{1,i}}{\delta_{1,i}^2 + \mu_i\delta_{2,i}} \quad \text{(Halley's correction)}; \end{aligned} \tag{8}$$

3° $S_{\lambda,i}(\boldsymbol{a},\boldsymbol{b}) = \sum_{j=1}^{i-1} \frac{\mu_j}{(z_i - a_j)^\lambda} + \sum_{j=i+1}^{\nu} \frac{\mu_j}{(z_i - b_j)^\lambda},$

$$f_i(\boldsymbol{a},\boldsymbol{b}) = \mu_i(\alpha+1)S_{2,i}(\boldsymbol{a},\boldsymbol{b}) - \alpha(\alpha+1)S_{1,i}^2(\boldsymbol{a},\boldsymbol{b}),$$

where $\boldsymbol{a} = (a_1,\dots,a_\nu)$ and $\boldsymbol{b} = (b_1,\dots,b_\nu)$ are some vectors of distinct complex numbers. If $\boldsymbol{a} = \boldsymbol{b} = \boldsymbol{z} = (z_1,\dots,z_\nu)$, then we will write $S_{\lambda,i}(\boldsymbol{z},\boldsymbol{z}) = S_{\lambda,i}$ and $f_i(\boldsymbol{z},\boldsymbol{z}) = f_i$ as in (1) and (2). We note that $N(z_i) = \mu_i P(z_i)/P'(z_i)$ is a correction of Newton's type introduced by Schröder [8]. Such a notation will be kept throughout this paper.

4° Types of approximations:

$\boldsymbol{z} = (z_1,\dots,z_\nu)$ (current approximations),

$\hat{\boldsymbol{z}} = (\hat{z}_1,\dots,\hat{z}_\nu)$ (new approximations),

$\boldsymbol{z}_N = (z_{N,1},\dots,z_{N,\nu}),\quad z_{N,i} = z_i - N(z_i)$ (Schröder's approximations),

$\boldsymbol{z}_H = (z_{H,1},\dots,z_{H,\nu}),\quad z_{H,i} = z_i - H(z_i)$ (Halley's approximations).

The correction terms (7) and (8) appear in the well known iterative formulas

$\hat{z} = z - N(z)$ (Schröder's method), and $\hat{z} = z - H(z)$ (Halley's method),

for finding a multiple zero, having respectively quadratic and cubic convergence.

5° The abbreviations TS and SS stand for the **t**otal-**s**tep methods ("Jacobi" or parallel mode) and **s**ingle-**s**tep methods (serial or "Gauss-Seidel" mode).

Taking certain approximations z_j of ζ_j on the right-hand side of the fixed point relation (6), on the left side we will obtain a new approximation $\hat{z}_i$ to the zero ζ_i instead of ζ_i. In this way we are able to construct some modified iterative processes for the simultaneous determination of multiple zeros of a polynomial.

First, for $i \in I_\nu$, we will state total-step methods:

Basic total-step method (TS)*:*

$$\hat{z}_i = z_i - \frac{\mu_i(\alpha+1)}{\alpha\delta_{1,i} + \Big[\mu_i(\alpha+1)\delta_{2,i} - \alpha\delta_{1,i}^2 - f_i(\boldsymbol{z},\boldsymbol{z})\Big]_*^{1/2}}. \tag{TS}$$

Total-step method with Newton's correction (TSN)*:*

$$\hat{z}_i = z_i - \frac{\mu_i(\alpha+1)}{\alpha\delta_{1,i} + \Big[\mu_i(\alpha+1)\delta_{2,i} - \alpha\delta_{1,i}^2 - f_i(\boldsymbol{z}_N,\boldsymbol{z}_N)\Big]_*^{1/2}}. \tag{TSN}$$

Total-step method with Halley's correction (TSH)*:*

$$\hat{z}_i = z_i - \frac{\mu_i(\alpha+1)}{\alpha\delta_{1,i} + \Big[\mu_i(\alpha+1)\delta_{2,i} - \alpha\delta_{1,i}^2 - f_i(\boldsymbol{z}_H,\boldsymbol{z}_H)\Big]_*^{1/2}}. \tag{TSH}$$

The order of convergence of each of the three aforementioned total-step methods can be increased using any new approximation as soon as it is calculated. In this manner, for $i \in I_\nu$, we construct the following single-step methods for multiple zeros:

Basic single-step method (SS)*:*

$$\hat{z}_i = z_i - \frac{\mu_i(\alpha+1)}{\alpha\delta_{1,i} + \Big[\mu_i(\alpha+1)\delta_{2,i} - \alpha\delta_{1,i}^2 - f_i(\hat{\boldsymbol{z}}, \boldsymbol{z})\Big]_*^{1/2}}. \tag{SS}$$

Single-step method with Newton's correction (SSN)*:*

$$\hat{z}_i = z_i - \frac{\mu_i(\alpha+1)}{\alpha\delta_{1,i} + \Big[\mu_i(\alpha+1)\delta_{2,i} - \alpha\delta_{1,i}^2 - f_i(\hat{\boldsymbol{z}}, \boldsymbol{z}_N)\Big]_*^{1/2}}. \tag{SSN}$$

Single-step method with Halley's correction (SSH)*:*

$$\hat{z}_i = z_i - \frac{\mu_i(\alpha+1)}{\alpha\delta_{1,i} + \Big[\mu_i(\alpha+1)\delta_{2,i} - \alpha\delta_{1,i}^2 - f_i(\hat{\boldsymbol{z}}, \boldsymbol{z}_H)\Big]_*^{1/2}}. \tag{SSH}$$

Remark 1. It is assumed that two values of the (complex) square root have to be taken in the above iterative formulas. Considering the fixed point relation (6) and all modified methods of presented family given above, we observe that a "proper" sign in front of the square root has to be chosen. We choose the sign so that a smaller step $|\hat{z}_i - z_i|$ is taken. We use the symbol $*$ in the above families of methods to indicate the choice of the proper value of the square root involved in the presented iterative formulas.

For some specific values of the parameter α, from the families of methods listed above we obtain special cases of these families as *Ostrowski-like method* ($\alpha = 0$), *Laguerre-like method* ($\alpha = \mu_i/(n-\mu_i)$), *Euler-like method* ($\alpha = 1$) and *Halley-like method* ($\alpha = -1$). The names come from the similarity with the quoted classical methods. Indeed, omitting the sums $S_{1,i}$ and $S_{2,i}$ in the above formulas, we obtain the corresponding well known classical methods.

2. Convergence analysis

Studying the convergence analysis of the total-step methods (TS), (TSN) and (TSH), we will consider all three methods simultaneously. The same is valid for the single-step methods (SS), (SSN) and (SSH). For this purpose we denote these methods with the additional indices 1 (for (TS) and (SS)), 2 (for (TSN) and (SSN)) and 3 (for (TSH) and (SSH)), and, in the same manner, we denote the corresponding vectors of approximations as follows:

$$\begin{aligned}
\boldsymbol{z}^{(1)} &= \boldsymbol{z} = (z_1, \ldots, z_\nu),\\
\boldsymbol{z}^{(2)} &= \boldsymbol{z}_N = (z_{N,1}, \ldots, z_{N,\nu}),\\
\boldsymbol{z}^{(3)} &= \boldsymbol{z}_H = (z_{H,1}, \ldots, z_{H,\nu}).
\end{aligned}$$

Now we are in possibility to rewrite all the mentioned total-step methods, denoted with (TS(k)) ($k = 1, 2, 3$), in the unique form as

$$\hat{z}_i = z_i - \frac{\mu_i(\alpha+1)}{\alpha\delta_{1,i} + \Big[\mu_i(\alpha+1)\delta_{2,i} - \alpha\delta_{1,i}^2 - f_i(\boldsymbol{z}^{(k)}, \boldsymbol{z}^{(k)})\Big]_*^{1/2}}, \qquad \text{(TS(k))}$$

for $i \in I_\nu$, $k = 1, 2, 3$ and $\alpha \neq -1$. Using the above notation for the arguments of f_i, the single-step methods (SS), (SSN) and (SSH), denoted commonly with (SS(k)), can be written in the unique form

$$\hat{z}_i = z_i - \frac{\mu_i(\alpha+1)}{\alpha\delta_{1,i} + \Big[\mu_i(\alpha+1)\delta_{2,i} - \alpha\delta_{1,i}^2 - f_i(\hat{\boldsymbol{z}}, \boldsymbol{z}^{(k)})\Big]_*^{1/2}} \qquad \text{(SS(k))}$$

where $i \in I_\nu$, $k = 1, 2, 3$ and $\alpha \neq -1$.

In a particular case $\alpha = \mu_i/(n-\mu_i)$, from the iterative formulas (TS(k)) and (SS(k)) we obtain Laguerre-like methods considered in [6].

We will always assume that $\alpha \neq -1$ in all iterative formulas presented above. If $\alpha = -1$, then, applying a limiting operation we obtain the methods of Halley's type

$$\hat{z}_i = z_i - \frac{2\mu_i\delta_{1,i}}{\delta_{1,i}^2 + \mu_i\delta_{2,i} - S_{1,i}^2(\boldsymbol{z}^{(k)}, \boldsymbol{z}^{(k)}) - \mu_i S_{2,i}(\boldsymbol{z}^{(k)}, \boldsymbol{z}^{(k)})},$$

for $i \in I_\nu$ and $k = 1, 2, 3$, whose basic variant and some improvements were considered in [9].

To avoid any confusion, we emphasize that, in the situation when the iteration index is omitted, the superscript k denotes the corresponding method. Following this notation we introduce the corrections $\Delta_{k,i}$ ($k = 1, 2, 3$) by

$$\Delta_{1,i} = 0, \quad \Delta_{2,i} = N_i, \quad \Delta_{3,i} = H_i.$$

Let us introduce the notations

$$d = \min_{\substack{i,j \\ i \neq j}} |\zeta_i - \zeta_j|, \quad q = \frac{4n}{d}$$

and assume that the conditions

$$(9) \qquad |\varepsilon_i| < \frac{d}{4n} = \frac{1}{q} \quad (i = 1, \ldots, \nu)$$

are fulfilled. In what follows, we will always assume that $2 \leq \nu \leq n$, $n \geq 3$ and $\alpha \neq -1$. Also, in our convergence analysis we will deal with the parameter α lying in the disk $|z| < 2.4$ centered at the origin (that is, $|\alpha| < 2.4$).

Lemma 1 *Let $z_1 \ldots, z_\nu$ be distinct approximations to the zeros $\zeta_1, \ldots, \zeta_\nu$, and let $\varepsilon_i = z_i - \zeta_i$, $\hat{\varepsilon}_i = \hat{z}_i - \zeta_i$, where $\hat{z}_1, \ldots, \hat{z}_\nu$ are approximations produced by the iterative methods* TS(k). *If* (9) *holds and $|\alpha| < 2.4 \wedge \alpha \neq -1$, then*

$(i)\quad |\hat{\varepsilon}_i| \le \dfrac{q^{k+2}}{n-1}|\varepsilon_i|^3 \sum_{j\ne i} \mu_j |\varepsilon_j|^k \quad (i \in I_\nu;\ k = 1,2,3);$

$(ii)\quad |\hat{\varepsilon}_i| < \dfrac{d}{4n} = \dfrac{1}{q} \quad (i = 1,\dots,\nu).$

The proof of Lemma 1 is extensive and tedious but elementary, and will be omitted to save space. The reader interested in this providing technique may find it in [6]

Assume that $z_1^{(0)},\dots,z_\nu^{(0)}$ are reasonably close approximations to the zeros $\zeta_1,\dots,\zeta_\nu$ of the polynomial P, and let

$$\varepsilon_i^{(m)} = z_i^{(m)} - \zeta_i, \quad \varepsilon^{(m)} = \max_{1\le i\le \nu} |\varepsilon_i^{(m)}|,$$

where $z_1^{(m)},\dots,z_\nu^{(m)}$ are approximations produced in the mth iterative step. Suppose that the initial conditions

$$|\varepsilon_i^{(0)}| < \frac{d}{4n} = \frac{1}{q} \quad (i = 1,\dots,\nu) \tag{10}$$

hold. Now, using the results presented in the previous section we will estimate the convergence rate of the total-step methods (TS(k)). The proof is derived under the condition (10), which means that a local convergence of these methods is assumed.

Theorem 1. *If the conditions* (10) *hold, the total-step methods* (TS(k)) *are convergent with the convergence order equal to* $k+3$ ($k = 1,2,3$).

Proof. The proof goes by induction. The relation (i) of Lemma 1 was derived under the condition (9). In the same manner, assuming that the condition (10) is valid, we estimate

$$|\varepsilon_i^{(1)}| \le \frac{q^{k+2}}{n-1}|\varepsilon_i^{(0)}|^3 \sum_{j\ne i} \mu_j |\varepsilon_j^{(0)}|^k < \frac{1}{q} \quad (i \in I_\nu;\ k = 1,2,3).$$

This means that we have that the implication

$$|\varepsilon_i^{(0)}| < \frac{d}{4n} = \frac{1}{q} \quad \Rightarrow \quad |\varepsilon_i^{(1)}| < \frac{d}{4n} = \frac{1}{q}$$

is valid. Now, we will prove that the condition (10) implies

$$|\varepsilon_i^{(m+1)}| < \frac{q^{k+2}}{n-1}|\varepsilon_i^{(m)}|^3 \sum_{j\ne i} \mu_j |\varepsilon_j^{(m)}|^k < \frac{1}{q} \tag{11}$$

for each $m = 0,1,\dots$, $k = 1,2,3$ and $i = 1,\dots,\nu$. Substituting $|\varepsilon_i^{(m)}| = t_i^{(m)}/q$ in (11), we obtain

$$t_i^{(m+1)} < \frac{\big(t_i^{(m)}\big)^3}{n-1} \sum_{j\ne i} \mu_j \big(t_j^{(m)}\big)^k \quad (i \in I_\nu;\ k = 1,2,3). \tag{12}$$

Taking $t^{(m)} = \max_{1 \le i \le \nu} t_i^{(m)}$, then by (10) we get

$$q|\varepsilon_i^{(0)}| = t_i^{(0)} \le t^{(0)} < 1, \quad (i \in I_\nu).$$

In regard to the last inequality and from (12) we obtain $t_i^{(m)} < 1$ for all $i = 1, \ldots, \nu$ and $m = 1, 2, \ldots$. According to this we obtain from (12)

$$t_i^{(m+1)} < \frac{\big(t_i^{(m)}\big)^3}{n-1}(n-\mu_i)\big(t^{(m)}\big)^k \le \big(t^{(m)}\big)^{k+3}, \tag{13}$$

$(k = 1, 2, 3)$, and conclude that the sequences $\{t_i^{(m)}\}$ $(i \in I_\nu)$ converge to 0. Consequently, the sequences $\{|\varepsilon_i^{(m)}|\}$ are also convergent, which means that $z_i^{(m)} \to \zeta_i$ $(i \in I_\nu)$. Finally, from (13) we may conclude that the total-step methods (TS(k)) have the convergence order $k+3$, that is, the total-step methods (TS) $(k = 1)$, (TSN) $(k = 2)$ and (TSH) $(k = 3)$ have the order of convergence *four*, *five* and *six*, respectively. □

Let us consider now the convergence rate of the single-step methods. Starting from the initial conditions (10), we can prove that the inequalities

$$\begin{aligned} |\varepsilon_i^{(m+1)}| &< \frac{q^{k+2}}{n-1}|\varepsilon_i^{(m)}|^3 \left(\sum_{j=1}^{i-1} \mu_j |\varepsilon_j^{(m+1)}| + q^{k-1} \sum_{j=i+1}^{\nu} \mu_j |\varepsilon_j^{(m)}|^k \right) \\ &< \frac{1}{q}, \quad (k = 1, 2, 3) \end{aligned} \tag{14}$$

hold for each $m = 0, 1, \ldots$ and $i = 1, \ldots, \nu$, assuming that for $i = 1$ the first sum in (14) is neglected. Substituting $|\varepsilon_i^{(m)}| = t_i^{(m)}/q$ in (14), we obtain

$$t_i^{(m+1)} < \frac{\big(t_i^{(m)}\big)^3}{n-1} \left(\sum_{j=1}^{i-1} \mu_j t_j^{(m+1)} + \sum_{j=i+1}^{\nu} \mu_j \big(t_j^{(m)}\big)^k \right), \tag{15}$$

for $i \in I_\nu$ and $k = 1, 2, 3$. The derivation of the inequalities (14) is essentially the same as that given above for the total-step methods. For this reason our main attention will be devoted to the precise estimation of the convergence rate of the single-step methods (SS(k)). This convergence analysis, similar to that presented by Alefeld and Herzberger [1] (see, also, [5]), uses the notion of the R-order of convergence introduced by Ortega and Rheinboldt [4]. The R-order of an iterative method IM with the limit point $\boldsymbol{\zeta}$ will be denoted by $O_R(\text{(IM)}, \boldsymbol{\zeta})$.

Theorem 2. *Assume that the initial conditions* (10) *hold. Then the R-order of convergence of the single-step methods* (SS(k))*, for which the relations* (14) *are valid, is given by*

$$O_R((SS(k)), \boldsymbol{\zeta}) \ge 3 + x_\nu(k),$$

where $x_\nu(k) > k$ is the unique positive root of the equation

$$x^\nu - k^{\nu-1}x - 3k^{\nu-1} = 0.$$

Proof. As in the proof of Theorem 1, we first note that from the inequality (10) it follows

$$|\varepsilon_i^{(0)}|q = t_i^{(0)} \le t = \max_{1\le i\le \nu} t_i^{(0)} < 1. \tag{16}$$

In view of this we conclude from (15) that the sequences $\{t_i^{(m)}\}$ and $\{|\varepsilon_i^{(m)}|\}$ tend to 0 for all $i = 1, \dots, \nu$, which means that $z_i^{(m)} \to \zeta_i$ $(i \in I_\nu)$.

Following Alefeld and Herzberger [1], from the relations (15) and (16) we derive

$$t_i^{(m+1)} \le t^{r_i^{(m)}} \quad (i = 1, \dots, \nu;\ m = 0, 1, \dots).$$

The column vectors $\boldsymbol{r}^{(m)} = [r_1^{(m)} \cdots r_\nu^{(m)}]^T$ are successively computed by

$$\boldsymbol{r}^{(m+1)} = A_\nu(k)\,\boldsymbol{r}^{(m)} \tag{17}$$

starting with $\boldsymbol{r}^{(0)} = [1 \cdots 1]^T$. The $\nu \times \nu$ matrix $A_\nu(k)$ in (17) is given by

$$A_\nu(k) = \begin{bmatrix} 3 & k & & & & & \\ & 3 & k & & \mathbf{O} & & \\ & & 3 & k & & & \\ & & & \ddots & \ddots & & \\ & & \mathbf{O} & & & & \\ & & & & & 3 & k \\ 3 & k & 0 & 0 & \cdots & 0 & 3 \end{bmatrix} \qquad (k = 1, 2, 3)$$

(see [5, Sec. 2.3] for the more general case). The characteristic polynomial of the matrix $A_\nu(k)$ is

$$g_\nu(\lambda) = (\lambda - 3)^\nu - (\lambda - 3)k^{\nu-1} - 3k^{\nu-1}.$$

Substituting $x = \lambda - 3$, we get

$$\phi_\nu(x) = g_\nu(x + 3) = x^\nu - k^{\nu-1}x - 3k^{\nu-1}.$$

The function $\phi_\nu(x)$ is monotonically increasing for $x > x_s = k/(\nu^{1/(\nu-1)})$ and decreasing for $x < x_s$. In addition,

$$\phi_\nu(x_s) < 0, \quad \phi_\nu(0) = \phi_\nu(k) = -3k^{\nu-1} < 0, \quad \phi_\nu(k+1) > 0.$$

From these data we conclude that the equation

$$x^\nu - k^{\nu-1}x - 3k^{\nu-1} = 0 \tag{18}$$

has a unique positive root in the interval $(k, k+1)$. The corresponding (positive) eigenvalue of the matrix $A_\nu(k)$ is $3 + x_\nu(k)$. The matrix $A_\nu(k)$ is irreducible and primitive, so that it has the unique positive eigenvalue equal to its spectral radius $\rho(A_\nu(k))$. According to the analysis presented in [1] it can be shown that the lower bound of the R-order of iterative method (SS(k)), for which the inequalities (15) are valid, is given by the spectral radius $\rho(A_\nu(k))$. Therefore, we have

$$O_R((\mathrm{SS}(k)), \boldsymbol{\zeta}) \geq \rho(A_\nu(k)) = 3 + x_\nu(k),$$

where $x_\nu(k) > k$ is the unique positive root of the equation (18). □

The lower bounds of $O_R((\mathrm{SS}), \boldsymbol{\zeta})$, $O_R((\mathrm{SSN}), \boldsymbol{\zeta})$ and $O_R((\mathrm{SSH}), \boldsymbol{\zeta})$, obtained by solving the equation (18), are tabulated for $\nu = 3(1)10$ in Table 1.

Methods \ ν	3	4	5	6	7	8	9	very large ν
(SS):	4.672	4.453	4.341	4.274	4.229	4.196	4.172	$\to 4$
(SSN):	5.862	5.586	5.443	5.357	5.299	5.257	5.225	$\to 5$
(SSH):	6.974	6.662	6.503	6.404	6.339	6.291	6.255	$\to 6$

Table 1 The lower bound of the R-order of convergence

3. Numerical results

To demonstrate the convergence speed of the proposed simultaneous methods, we tested a lot of polynomial equations. In this section we give some selected examples chosen among many numerical experiments. The corresponding algorithms were realized using the programming package *Mathematica* 5 on PC PENTIUM IV. In order to save all significant digits of the obtained approximations, we employed multiple precision arithmetic.

The proposed total step as well as single step methods with the Schröder and Halley corrections use the already calculated values P, P', P'' at the points $z_1, \ldots, z_\nu$ so that the convergence speed of the implemented iterative methods is accelerated with the negligible number of additional operations. In this manner a very high computational efficiency of the proposed methods is provided, which is the main advantage of the presented methods.

The performed numerical experiments demonstrated very fast convergence of the modified methods for finding multiple zeros. For illustration, we present two numerical examples. As a measure of closeness of approximations with regard to the exact zeros, we have calculated Euclid's norm

$$e^{(m)} := ||\boldsymbol{z}^{(m)} - \boldsymbol{\zeta}||_E = \left(\sum_{i=1}^{\nu} \mu_i \left| z_i^{(m)} - \zeta_i \right|^2 \right)^{1/2}.$$

Example 1 We applied the proposed methods (TS(k)) and (SS(k)), obtained for $\alpha = 0$, $\alpha = \mu_i/(n - \mu_i)$, $\alpha = 1/2$, $\alpha = 1$ and $\alpha = -1$, for the simultaneous approximation to the multiple zeros of the polynomial

$$\begin{aligned} P(z) &= z^{13}-(1-2i)z^{12}-(10+2i)z^{11}-(30+18i)z^{10}+(35-62i)z^{9} \\ &\quad +(293+52i)z^{8}+(452+524i)z^{7}-(340-956i)z^{6} \\ &\quad -(2505+156i)z^{5}-(3495+4054i)z^{4}-(538+7146i)z^{3} \\ &\quad +(2898-5130i)z^{2}+(2565-1350i)z+675 \\ &= (z+1)^{4}(z-3)^{3}(z+i)^{2}(z^{2}+2z+5)^{2}. \end{aligned}$$

The exact zeros of this polynomial are $\zeta_1=-1$, $\zeta_2=3$, $\zeta_3=-i$ and $\zeta_{4,5}=-1\pm 2i$ with the multiplicities $\mu_1=4$, $\mu_2=3$, $\mu_3=\mu_4=\mu_5=2$. The following complex numbers were chosen as starting approximations to these zeros:

$$\begin{aligned} &z_1^{(0)}=-0.7+0.3i, \quad z_2^{(0)}=2.7+0.3i, \quad z_3^{(0)}=0.3-0.8i, \\ &z_4^{(0)}=-1.2-2.3i, \quad z_5^{(0)}=-1.3+2.2i. \end{aligned}$$

In the presented example for the initial approximations we have $e^{(0)}\approx 1.43$. The measure of accuracy $e^{(m)}$ $(m=1,2,3)$ is displayed in Table 2.

Meth.		$\alpha=0$	$\alpha=\mu_i/(n-\mu_i)$	$\alpha=1/2$	$\alpha=1$	$\alpha=-1$
(TS)	$e^{(1)}$	2.39(−2)	1.62(−2)	1.93(−2)	6.32(−2)	5.72(−2)
	$e^{(2)}$	1.47(−8)	1.18(−9))	1.39(−9)	8.80(−7)	1.54(−6)
	$e^{(3)}$	8.08(−34)	6.08(−38)	9.63(−38)	4.96(−26)	2.20(−26)
(TSN)	$e^{(1)}$	7.64(−3)	7.26(−3)	7.24(−3)	7.35(−3)	8.61(−3)
	$e^{(2)}$	1.95(−13)	1.05(−13)	7.74(−14)	1.21(−13)	5.17(−13)
	$e^{(3)}$	2.72(−66)	8.04(−68)	1.01(−69)	1.40(−66)	9.97(−64)
(TSH)	$e^{(1)}$	1.94(−3)	1.66(−3)	1.70(−3)	5.20(−3)	3.32(−3)
	$e^{(2)}$	1.35(−19)	2.78(−20)	2.06(−20)	2.14(−17)	2.61(−17)
	$e^{(3)}$	1.69(−116)	7.16(−121)	6.04(−121)	2.46(−103)	1.88(−101)
(SS)	$e^{(1)}$	1.54(−2)	1.38(−2)	1.42(−2)	1.51(−2)	1.99(−2)
	$e^{(2)}$	3.48(−10)	1.95(−10)	2.54(−10)	1.03(−9)	2.02(−9)
	$e^{(3)}$	1.18(−42)	2.35(−43)	1.19(−41)	5.72(−40)	2.40(−38)
(SSN)	$e^{(1)}$	6.20(−3)	5.77(−3)	5.94(−3)	6.35(−3)	7.61(−3)
	$e^{(2)}$	1.82(−14)	1.20(−14)	1.95(−14)	6.98(−14)	1.28(−13)
	$e^{(3)}$	1.35(−77)	2.31(−78)	7.86(−74)	5.78(−70)	6.98(−70)
(SSN)	$e^{(1)}$	1.57(−3)	1.51(−3)	1.57(−3)	1.88(−3)	2.06(−3)
	$e^{(2)}$	1.49(−20)	9.35(−21)	1.49(−20)	1.11(−19)	1.86(−19)
	$e^{(3)}$	5.26(−133)	1.39(−134)	1.57(−126)	3.77(−118)	1.03(−119)

Table 2 Euclid's norm of errors; $A(-q)$ means $A\times 10^{-q}$.

Example 2 The same iterative methods as in Example 1 were applied for the simultaneous approximation to the zeros of the polynomial

$$\begin{aligned} P(z) &= z^{12}+(3-6i)z^{11}-(24+18i)z^{10}-(72-80i)z^{9} \\ &\quad +(230+240i)z^{8}+(690-612i)z^{7}-(1332+1836i)z^{6} \\ &\quad -(3996-2488i)z^{5}+(4225+7464i)z^{4}+(12675-6150i)z^{3} \\ &\quad -(7500+18450i)z^{2}-(22500-5000i)z+15000i \\ &= (z+3)(z-2i)^{3}(z^{2}+4z+5)^{2}(z^{2}-4z+5)^{2}. \end{aligned}$$

The exact zeros of this polynomial are $\zeta_1=-3$, $\zeta_2=2i$, $\zeta_{3,4}=-2\pm i$, $\zeta_{5,6}=2\pm i$, with the multiplicities $\mu_1=1$, $\mu_2=3$, $\mu_3=\mu_4=\mu_5=\mu_6=2$. All tested methods started with the following initial approximations:

$$z_1^{(0)} = -3.3 + 0.2i, \quad z_2^{(0)} = 0.3 + 2.3i, \quad z_3^{(0)} = -2.3 + 1.2i,$$
$$z_4^{(0)} = -2.3 - 1.3i, \quad z_5^{(0)} = 2.3 + 1.3i, \quad z_6^{(0)} = 2.3 - 1.2i.$$

In the presented example, for the initial approximations we have $e^{(0)} = 1.34$. The measure of accuracy $e^{(m)}$ $(m = 1, 2, 3)$ is displayed in Table 3.

Meth.		$\alpha = 0$	$\alpha = \mu_i/(n-\mu_i)$	$\alpha = 1/2$	$\alpha = 1$	$\alpha = -1$
(TS)	$e^{(1)}$	1.07(−2)	5.35(−3)	1.17(−2)	4.88(−2)	3.10(−2)
	$e^{(2)}$	1.39(−10)	5.09(−12)	1.93(−10)	8.13(−8)	1.92(−8)
	$e^{(3)}$	4.85(−43)	1.74(−48)	1.87(−41)	6.86(−31)	3.14(−33)
(TSN)	$e^{(1)}$	3.30(−3)	2.04(−3)	3.30(−3)	1.17(−2)	8.49(−3)
	$e^{(2)}$	5.61(−16)	7.11(−17)	1.56(−15)	1.42(−12)	1.38(−13)
	$e^{(3)}$	1.72(−79)	2.21(−84)	3.61(−77)	8.14(−62)	2.44(−67)
(TSH)	$e^{(1)}$	6.63(−4)	5.03(−4)	7.22(−4)	1.97(−3)	1.47(−3)
	$e^{(2)}$	1.79(−23)	4.91(−24)	4.28(−23)	5.08(−20)	2.42(−21)
	$e^{(3)}$	4.56(−141)	2.29(−144)	3.60(−138)	1.48(−119)	1.01(−127)
(SS)	$e^{(1)}$	5.57(−3)	3.96(−3)	6.83(−3)	1.96(−2)	1.48(−2)
	$e^{(2)}$	5.62(−13)	4.73(−13)	2.56(−12)	4.17(−10)	5.52(−10)
	$e^{(3)}$	6.91(−52)	1.53(−52)	5.30(−49)	4.86(−39)	1.57(−39)
(SSN)	$e^{(1)}$	2.33(−3)	1.53(−3)	2.17(−3)	6.26(−3)	5.54(−3)
	$e^{(2)}$	1.73(−15)	1.24(−17)	3.49(−17)	2.49(−14)	1.88(−14)
	$e^{(3)}$	8.51(−89)	1.47(−89)	1.08(−85)	1.45(−71)	2.16(−72)
(SSH)	$e^{(1)}$	6.17(−4)	4.45(−4)	5.30(−4)	1.37(−3)	1.28(−3)
	$e^{(2)}$	1.13(−23)	1.39(−24)	4.06(−24)	1.65(−21)	7.93(−22)
	$e^{(3)}$	1.12(−145)	1.11(−150)	5.24(−146)	9.08(−128)	4.89(−131)

Table 3 Euclid's norm of errors; $A(-q)$ means $A \times 10^{-q}$.

From Tables 2 and 3, and a number of tested polynomial equations we can conclude that the results obtained by the proposed methods coincide well with theoretical results given in Theorems 1 and 2. Also, we note that two iterative steps of the presented families of methods are usually sufficient in solving most of practical problems when initial approximations are reasonably good and polynomials are well-conditioned. The third iteration is given to demonstrate remarkably fast convergence and present approximations of very high accuracy, rarely needed in practice at present.

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