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The Duals of the 2-Modular Irreducible Modules of the Alternating Groups

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Abstract. We determine the dual modules of all irreducible modules of alternating groups over fields of characteristic 2.

Key words: symmetric group; alternating group; dual module; irreducible module; characteristic 2

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1 Introduction and statement of the result

Let $\mathcal{S}_n$ be the symmetric group of degree $n \geq 1$ and let k be a field of characteristic $p > 0$. In [7, Theorem 11.5] G. James constructed all irreducible $k\mathcal{S}_n$-modules D^λ where λ ranges over the p-regular partitions of n. Here a partition is p-regular if each of its parts occurs with multiplicity less than p.

As the alternating group $\mathcal{A}_n$ has index 2 in $\mathcal{S}_n$, the restriction $D^\lambda{\downarrow_{\mathcal{A}_n}}$ is either irreducible or splits as a direct sum of two non-isomorphic irreducible $k\mathcal{A}_n$-modules. Moreover, every irreducible $k\mathcal{A}_n$-module is a direct summand of some $D^\lambda{\downarrow_{\mathcal{A}_n}}$.

Henceforth we will assume, unless stated otherwise, that k is a field of characteristic 2 which is a splitting field for the alternating group $\mathcal{A}_n$. For this, it suffices that k contains the finite field $\mathbb{F}_4$. D. Benson [1] has classified all irreducible $k\mathcal{A}_n$-modules:

Proposition 1.1. *Let $\lambda = (\lambda_1 > \lambda_2 > \cdots > \lambda_{2s-1} > \lambda_{2s} \geq 0)$ be a strict partition of n. Then $D^\lambda{\downarrow_{\mathcal{A}_n}}$ is reducible if and only if*

(*i*) $\lambda_{2j-1} - \lambda_{2j} = 1$ *or* 2*, for* $j = 1, \ldots, s$*, and*

(*ii*) $\lambda_{2j-1} + \lambda_{2j} \not\equiv 2 \pmod 4$*, for* $j = 1, \ldots, s$.

In this note we determine the dual of each irreducible $k\mathcal{A}_n$-module. Now $D^\lambda{\downarrow_{\mathcal{A}_n}}$ is a self-dual $k\mathcal{A}_n$-module, as D^λ is a self-dual $k\mathcal{S}_n$-module. So we only need to determine the dual of an irreducible $k\mathcal{A}_n$-module which is a direct summand of $D^\lambda{\downarrow_{\mathcal{A}_n}}$, when this module is reducible.

Theorem 1.2. *Let λ be a strict partition of n such that $D^\lambda{\downarrow_{\mathcal{A}_n}}$ is reducible. Then the two irreducible direct summands of $D^\lambda{\downarrow_{\mathcal{A}_n}}$ are self-dual if $\sum_{j=1}^{s} \lambda_{2j}$ is even and are dual to each other if $\sum_{j=1}^{s} \lambda_{2j}$ is odd.*

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For example $D^{(7,5,1)}{\downarrow_{\mathcal{A}_{13}}} \cong S \oplus S^*$, for a non self-dual irreducible $k\mathcal{A}_{13}$-module S, and $D^{(5,4,3,1)}{\downarrow_{\mathcal{A}_{13}}}$ decomposes similarly. On the other hand $D^{(7,6)}{\downarrow_{\mathcal{A}_{13}}} \cong S_1 \oplus S_2$ where S_1 and S_2 are irreducible and self-dual.

In order to prove Theorem 1.2, we use the following elementary result, which requires the assumption that k has characteristic 2:

Lemma 1.3. *Let G be a finite group and let M be a semisimple kG-module which affords a non-degenerate G-invariant symmetric bilinear form B. Suppose that $B(tm, m) \neq 0$, for some involution $t \in G$ and some $m \in M$. Then M has a self-dual irreducible direct summand.*

Proof. We have $M = \bigoplus_{i=1}^n M_i$, for some $n \geq 1$ and irreducible kG-modules $M_1, \dots, M_n$. Write $m = \sum m_i$, with $m_i \in M_i$, for all i. Then

$$\begin{aligned} B(tm, m) &= \sum_{1\leq i\leq n} B(tm_i, m_i) + \sum_{1\leq i<j\leq n}^{n} \big(B(tm_i, m_j) + B(tm_j, m_i)\big) \\ &= \sum_{1\leq i\leq n} B(tm_i, m_i). \end{aligned}$$

The last equality follows from the fact that $\mathrm{char}(k) = 2$ and

$$B(tm_i, m_j) = B\big(m_i, t^{-1}m_j\big) = B(m_i, tm_j) = B(tm_j, m_i).$$

Without loss of generality $B(tm_1, m_1) \neq 0$. Then B restricts to a non-zero G-invariant symmetric bilinear form B_1 on M_1. As M_1 is irreducible, B_1 is non-degenerate. So M_1 is isomorphic to its kG-dual M_1^*. ■

2 Known results on the symmetric and alternating groups

2.1 The irreducible modules of the symmetric groups

We use the ideas and notation of [7]. In particular for each partition λ of n, James defines the Young diagram $[\lambda]$ of λ, and the notions of a λ-tableau and a λ-tabloid.

Fix a λ-tableau x. So x is a filling of $[\lambda]$ with the symbols $\{1, \dots, n\}$. The corresponding λ-tabloid is $\{x\} := \{\sigma(x) \mid \sigma \in R_x\}$, where R_x is the row stabilizer of x. We regard $\{x\}$ as an ordered set partition of $\{1, \dots, n\}$. The $\mathbb{Z}$-span of the λ-tabloids forms the $\mathbb{Z}\mathcal{S}_n$-lattice M^λ, and the set of λ-tabloids is an $\mathcal{S}_n$-invariant $\mathbb{Z}$-basis of M^λ.

Recall from [7, Section 4] that corresponding to each tableau x there is a polytabloid $e_x := \sum \mathrm{sgn}(\sigma)\{\sigma x\}$ in M^λ. Here σ ranges over the permutations in the column stabilizer C_x of the tableau x. The Specht lattice S^λ is defined to be the $\mathbb{Z}$-span of all λ-polytabloids. In particular S^λ is a $\mathbb{Z}\mathcal{S}_n$-sublattice of M^λ; it has as $\mathbb{Z}$-basis the polytabloids corresponding to the standard λ-tableaux (i.e., the numbers increase from left-to-right along rows, and from top-to-bottom along columns).

Now James defines $\langle\,,\,\rangle$ to be the symmetric bilinear form on M^λ which makes the tabloids into an orthonormal basis. As the tabloids are permuted by the action of $\mathcal{S}_n$, it is clear that $\langle\,,\,\rangle$ is $\mathcal{S}_n$-invariant.

Suppose now that λ is a strict partition and consider the unique permutation $\tau \in R_x$ which reverses the order of the symbols in each row of the tableau x. In [7, Lemma 10.4] James shows that $\langle \tau e_x, e_x\rangle = 1$, as $\{x\}$ is the only tabloid common to e_x and $e_{\tau x}$ (in fact James proves that $\langle \tau e_x, e_x\rangle$ is coprime to p, if λ is p-regular, for some prime p). Set $J^\lambda := \{x \in S^\lambda \mid \langle x, y\rangle \in 2\mathbb{Z}$, for all $y \in S^\lambda\}$. Then $2S^\lambda \subseteq J^\lambda$ and it follows from [7, Theorem 4.9] that $D^\lambda := (S^\lambda/J^\lambda) \otimes_{\mathbb{F}_2} k$ is an absolutely irreducible $k\mathcal{S}_n$-module, for any field k of characteristic 2.

2.2 The real 2-regular conjugacy classes of the alternating groups

A conjugacy class of a finite group G is said to be 2-regular if its elements have odd order. R. Brauer proved that the number of irreducible kG-modules equals the number of 2-regular conjugacy classes of G [4]. Now Brauer's permutation lemma holds for arbitrary fields [3, footnote 19]. So it is clear that the number of self-dual irreducible kG-modules equals the number of real 2-regular conjugacy classes of G.

We review some well known facts about the 2-regular conjugacy classes of the alternating group. See for example [8, Section 2.5].

Corresponding to each partition μ of n there is a conjugacy class C_μ of $\mathcal{S}_n$; its elements consist of all permutations of n whose orbits on $\{1,\dots,n\}$ have sizes $\{\mu_1,\dots,\mu_\ell\}$ (as multiset). So C_μ is 2-regular if and only if each μ_i is odd.

Let μ be a partition of n into odd parts. Then $C_\mu \subseteq \mathcal{A}_n$. If μ has repeated parts then C_μ is a conjugacy class of $\mathcal{A}_n$. As C_μ is closed under taking inverses, C_μ is a real conjugacy class of $\mathcal{A}_n$.

Now assume that μ has distinct parts. Then C_μ is a union of two conjugacy classes $C_\mu^\pm$ of $\mathcal{A}_n$. Set $m := \frac{n-\ell(\mu)}{2}$ and let $z \in C_\mu$. Then z is inverted by an involution $t \in \mathcal{S}_n$ of cycle type $(2^m, 1^{n-2m})$. Since $\mathrm{C}_{\mathcal{S}_n}(z) \cong \prod \mathbb{Z}/\mu_j\mathbb{Z}$ is odd, t generates a Sylow 2-subgroup of the extended centralizer $\mathrm{C}^*_{\mathcal{S}_n}(z)$ of z in $\mathcal{S}_n$. It follows that z is conjugate to z^{-1} in $\mathcal{A}_n$ if and only if $t \in \mathcal{A}_n$. This shows that $C_\mu^\pm$ are real classes of $\mathcal{A}_n$ if and only if $\frac{n-\ell(\mu)}{2}$ is even. This and the discussion above shows:

Lemma 2.1. *The number of self-dual irreducible $k\mathcal{A}_n$-modules equals the number of non-strict odd partitions of n plus twice the number of strict odd partitions μ of n for which $\frac{n-\ell(\mu)}{2}$ is even.*

3 Bressoud's bijection

We need a special case of a partition identity of I. Schur [9]. This was already used by Benson in his proof of Proposition 1.1:

Proposition 3.1 (Schur, 1926)**.** *The number of strict partitions of n into odd parts equals the number of strict partitions of n into parts congruent to $0, \pm 1 \pmod 4$ where consecutive parts differ by at least 4 and consecutive even parts differ by at least 8.*

D. Bressoud [5] has constructed a bijection between the relevant sets of partitions. We describe a simplified version of this bijection.

Let $\mu = (\mu_1 > \mu_2 > \cdots > \mu_\ell)$ be a strict partition of n whose parts are all odd. We subdivide μ into 'blocks' of at most two parts, working recursively from largest to smallest parts. Let $j \geq 1$ and suppose that $\mu_1, \mu_2, \dots, \mu_{j-1}$ have already been assigned to blocks. We form the block $\{\mu_j, \mu_{j+1}\}$ if $\mu_j = \mu_{j+1} + 2$, and the block $\{\mu_j\}$ otherwise (if $\mu_j \geq \mu_{j+1} + 4$). Let s be the number of resulting blocks of μ.

Next we form the sequence of positive integers $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_s)$, where σ_j is the sum of the parts in the j-th block of μ. Then the σ_j are distinct, as the odd parts form a decreasing sequence, with minimal difference 4, and the even parts form a decreasing sequence, with minimal difference 8. Moreover, each even σ_j is the sum of a pair of consecutive odd integers. So $\sigma_j \not\equiv 2 \pmod 4$, for all $j > 0$.

We get a composition ζ of $n + 2s(s-1)$ by defining

$$\zeta_1 = \sigma_1,\ \zeta_2 = \sigma_2 + 4,\ \dots,\ \zeta_s = \sigma_s + 4(s-1).$$

The even ζ_j form a decreasing sequence, with minimal difference 4, and the odd ζ_j form a weakly decreasing sequence ($\zeta_j = \zeta_{j+1}$ if and only if ζ_j, ζ_{j+1} represent two singleton blocks $\{2k-1\}$ and $\{2k-5\}$ of μ, for some $k \geq 0$).

Choose a permutation τ such that $\zeta_{\tau 1} \geq \zeta_{\tau 2} \geq \cdots \geq \zeta_{\tau s}$. Then we get a strict partition γ of n by defining

$$\gamma_1 = \zeta_{\tau 1},\ \gamma_2 = \zeta_{\tau 2} - 4,\ \ldots,\ \gamma_s = \zeta_{\tau s} - 4(s-1).$$

By construction, the minimal difference between the parts of γ is 4 and the minimal difference between the even parts of γ is 8. Moreover, $\gamma_j \equiv \zeta_{\tau j} \pmod 4$. So $\gamma_j \not\equiv 2 \pmod 4$. Then $\mu \to \gamma$ is Bressoud's bijection.

Finally form a strict partition λ of n which has $2s-1$ or $2s$ parts, by defining

$$(\lambda_{2j-1}, \lambda_{2j}) = \begin{cases} \left(\frac{\gamma_j}{2}+1, \frac{\gamma_j}{2}-1\right), & \text{if } \gamma_j \text{ is even or} \\ \left(\frac{\gamma_j+1}{2}, \frac{\gamma_j-1}{2}\right), & \text{if } \gamma_j \text{ is odd.} \end{cases}$$

Then λ satisfies the constraints (i) and (ii) of Proposition 1.1. Conversely, it is easy to see that if λ satisfies these constraints, then λ is the image of some strict odd partition μ of n under the above sequence of operations.

Lemma 3.2. *Let μ be a strict-odd partition of n and let λ be the strict partition of n constructed from μ as above. Then $\frac{n-\ell(\mu)}{2} = \sum \lambda_{2j}$.*

Proof. Each pair of consecutive parts λ_{2j-1}, λ_{2j} of λ corresponds to a block $\mathcal{B}$ of μ. Moreover by our description of Bressoud's bijection, there are integers $q_1, \ldots, q_s$, with $\sum q_j = 0$ such that

$$(\lambda_{2j-1} + 2q_j, \lambda_{2j} + 2q_j) = \begin{cases} \left(\frac{\mu_i+1}{2}, \frac{\mu_i-1}{2}\right), & \text{if } \mathcal{B} = \{\mu_i\}, \\ (\mu_i, \mu_{i+1}), & \text{if } \mathcal{B} = \{\mu_i, \mu_{i+1}\}. \end{cases}$$

In case $\mathcal{B} = \{\mu_i, \mu_{i+1}\}$, we have $\mu_i = \mu_{i+1} + 2$ and thus $\frac{\mu_i - 1}{2} + \frac{\mu_{i+1}-1}{2} = \lambda_{2j} + 2q_j$. We conclude that

$$\frac{n-\ell(\mu)}{2} = \sum_{i=1}^{\ell(\mu)} \frac{\mu_i - 1}{2} = \sum_{j=1}^{s} (\lambda_{2j} + 2q_j) = \sum_{j=1}^{s} \lambda_{2j}. \qquad \blacksquare$$

4 Proof of Theorem 1.2

Let $\mathrm{D}(n)$ be the set of strict partitions of n and let $\mathrm{S}(n)$ be the set of strict partitions of n which satisfy conditions (i) and (ii) in Proposition 1.1. So there are $2|\mathrm{S}(n)| + |\mathrm{D}(n) \backslash \mathrm{S}(n)|$ irreducible $k\mathcal{A}_n$-modules.

Next set $\mathrm{S}(n)^+ := \{\lambda \in \mathrm{S}(n) \mid \sum \lambda_{2j} \text{ is even}\}$. Then it follows from Lemmas 2.1 and 3.2 that the number of self-dual irreducible $k\mathcal{A}_n$-modules equals $2|\mathrm{S}(n)^+| + |\mathrm{D}(n) \backslash \mathrm{S}(n)|$. Now $D^\lambda{\downarrow_{\mathcal{A}_n}}$ is an irreducible self-dual $k\mathcal{A}_n$-module, for $\lambda \in \mathrm{D}(n) \backslash \mathrm{S}(n)$. So we can prove Theorem 1.2 by showing that the irreducible direct summands of $D^\lambda{\downarrow_{\mathcal{A}_n}}$ are self-dual for all $\lambda \in \mathrm{S}(n)^+$.

Suppose then that $\lambda \in \mathrm{S}(n)^+$. Let $\tau \in \mathcal{S}_n$ be the permutation which reverses each row of a λ-tableau, as discussed in Section 2.1. We claim that $\tau \in \mathcal{A}_n$. For τ is a product of $\sum_{i=1}^{2s} \lfloor \frac{\lambda_j}{2} \rfloor$ commuting transpositions. Now $\lfloor \frac{\lambda_{2j-1}}{2} \rfloor + \lfloor \frac{\lambda_{2j}}{2} \rfloor = \lambda_{2j}$, as $\lambda_{2j-1} - \lambda_{2j} = 1$, or $\lambda_{2j-1} - \lambda_{2j} = 2$ and both λ_{2j-1} and λ_{2j} are odd. So $\sum_{i=1}^{2s} \lfloor \frac{\lambda_i}{2} \rfloor = \sum_{j=1}^{s} \lambda_{2j}$ is even. This proves the claim.

Since D^λ is irreducible and the form $\langle\, , \rangle$ is non-zero, $\langle\, , \rangle$ is non-degenerate on D^λ. Write $D^\lambda{\downarrow_{\mathcal{A}_n}} = S_1 \oplus S_2$, where S_1 and S_2 are non-isomorphic irreducible modules. As $\tau \in \mathcal{A}_n$, it follows from Lemma 1.3 that we may assume that S_1 is self-dual. Now $S_2^* \not\cong S_1^* \cong S_1$ and S_2^* is isomorphic to a direct summand of $D^\lambda{\downarrow_{\mathcal{A}_n}}$. So S_2 is also self-dual. This completes the proof of the theorem.

5 Irreducible modules of alternating groups over fields of odd characteristic

We now comment briefly on what happens when k is a splitting field for $\mathcal{A}_n$ which has odd characteristic p. Let sgn be the sign representation of $k\mathcal{S}_n$. So sgn is 1-dimensional but non-trivial. G. Mullineux defined a bijection $\lambda \to \lambda^M$ on the p-regular partitions of n and conjectured that $D^\lambda \otimes \mathrm{sgn} = D^{\lambda^M}$ for all p-regular partitions λ of n. This was only proved in the 1990's by Kleshchev and Ford–Kleshchev. See [6] for details.

Now $D^\lambda{\downarrow_{\mathcal{A}_n}} \cong D^{\lambda^M}{\downarrow_{\mathcal{A}_n}}$, and $D^\lambda{\downarrow_{\mathcal{A}_n}}$ is irreducible if and only if $\lambda \neq \lambda^M$ See [2] for details. Moreover D^λ and D^{λ^M} are duals of each other, by [7, Theorem 6.6]. So $D^\lambda{\downarrow_{\mathcal{A}_n}}$ is self-dual, if $\lambda \neq \lambda^M$. However when $\lambda = \lambda^M$, we do not know how to determine when the two irreducible direct summands of $D^\lambda{\downarrow_{\mathcal{A}_n}}$ are self-dual.

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