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Discrete Orthogonal Polynomials with Hypergeometric Weights and Painlevé VI

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Abstract. We investigate the recurrence coefficients of discrete orthogonal polynomials on the non-negative integers with hypergeometric weights and show that they satisfy a system of non-linear difference equations and a non-linear second order differential equation in one of the parameters of the weights. The non-linear difference equations form a pair of discrete Painlevé equations and the differential equation is the σ-form of the sixth Painlevé equation. We briefly investigate the asymptotic behavior of the recurrence coefficients as $n \to \infty$ using the discrete Painlevé equations.

Key words: discrete orthogonal polynomials; hypergeometric weights; discrete Painlevé equations; Painlevé VI

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1 Introduction

In the past few years many semi-classical orthogonal polynomials were investigated and discrete and continuous Painlevé equations were found for their recurrence coefficients. In this paper we are interested in some discrete orthogonal polynomials on the integers $\mathbb{N} = \{0, 1, 2, 3, \dots\}$. For the orthonormal polynomials one has

$$\sum_{k=0}^{\infty} p_n(k) p_m(k) w_k = \delta_{m,n}, \qquad p_n(x) = \gamma_n x^n + \cdots$$

and the three term recurrence relation is

$$x p_n(x) = a_{n+1} p_{n+1}(x) + b_n p_n(x) + a_n p_{n-1}(x),$$

where, as usual, we take $a_0 = 0$, and the weights are such that all the moments are finite

$$m_n = \sum_{k=0}^{\infty} k^n w_k < \infty, \qquad n = 0, 1, 2, \dots.$$

For the monic orthogonal polynomials $P_n = p_n/\gamma_n$ the recurrence relation becomes

$$x P_n(x) = P_{n+1}(x) + b_n P_n(x) + a_n^2 P_{n-1}(x).$$

The following families have already been analyzed earlier:

This paper is a contribution to the Special Issue on Painlevé Equations and Applications in Memory of Andrei Kapaev. The full collection is available at https://www.emis.de/journals/SIGMA/Kapaev.html

- The Charlier polynomials $C_n(x;a)$ $(a>0)$ [4, Section VI.1], [18, Section 9.14] form a system of classical orthogonal polynomials on the integers $\mathbb{N}$ satisfying

$$\sum_{k=0}^{\infty} C_n(k;a)C_m(k;a)\frac{a^k}{k!}=0, \qquad n\neq m.$$

The monic Charlier polynomials $P_n(x)=(-1)^n a^n C_n(x;a)$ satisfy the recurrence relation

$$xP_n(x)=P_{n+1}(x)+(n+a)P_n(x)+naP_{n-1}(x),$$

hence the weights are $w_k=a^k/k!$ and the recurrence coefficients are $a_n^2=na$ and $b_n=n+a$. These are simple polynomial expressions in n and a.

- The Meixner polynomials $M_n(x;\beta,c)$ $(\beta>0,\ 0<c<1)$ [4, Section VI.3], [18, Section 9.10] are also a family of classical orthogonal polynomials:

$$\sum_{k=0}^{\infty} M_n(k;\beta,c)M_m(k;\beta,c)\frac{(\beta)_k c^k}{k!}=0, \qquad n\neq m,$$

and the recurrence coefficients are again simple and given by

$$a_n^2=\frac{n(n+\beta-1)c}{(1-c)^2}, \qquad b_n=\frac{n+(n+\beta)c}{1-c}. \tag{1.1}$$

When $\beta=-N$ is a negative integer and $c=\frac{p}{p-1}$, with $0<p<1$, then one finds Krawtchouk polynomials $K_n(x;p,N)$. This is a finite family of polynomials which are orthogonal for the binomial distribution. The recurrence coefficients are

$$a_n^2=np(1-p)(N+1-n), \qquad b_n=p(N-n)+n(1-p).$$

Note that $a_{N+1}^2=0$, which comes from the fact that this is a finite family of orthogonal polynomials with a measure supported on $N+1$ points.

- Generalized Charlier polynomials with weights $w_k=\frac{a^k}{k!(\beta)_k}$ $(a>0,\ \beta>0)$ were, for $\beta=1$, first considered in [14] and analyzed in [24]. The general case $\beta>0$ was investigated in [22] where the discrete Painlevé equations are given, and [11] where the Painlevé differential equation was given. Clarkson [6] found the connection with the Painlevé equation in a different way, starting from the Hankel determinants and the special function solutions of Painlevé equations. For $\beta=1$ the recurrence coefficients are given by $a_n^2=a\big(1-c_n^2\big)$ and $b_n=n+\sqrt{a}c_nc_{n+1}$, where c_n satisfies the discrete Painlevé II equation

$$c_{n+1}+c_{n-1}=\frac{nc_n}{\sqrt{a}(1-c_n^2)},$$

with initial conditions $c_0=1$ and $c_1=I_1\big(2\sqrt{a}\big)/I_0\big(2\sqrt{a}\big)$, where I_ν is the modified Bessel function. For $\beta\neq 1$ the recurrence coefficients satisfy

$$\big(a_{n+1}^2-a\big)\big(a_n^2-a\big)=a(b_n-n)(b_n-n+\beta-1),$$
$$b_n+b_{n-1}-n+\beta=\frac{an}{a_n^2},$$

with initial conditions $a_0^2=0$ and $b_0=\sqrt{a}I_\beta\big(2\sqrt{a}\big)/I_{\beta-1}\big(2\sqrt{a}\big)$. This is a limiting case of a discrete Painlevé IV equation with surface/symmetry $D_4^{(1)}$ [17, Section 8.1.16]. In [11, Theorem 2.1] it was also shown that b_n, as a function of the parameter a, satisfies a Painlevé V equation with parameter $\delta=0$. Such a Painlevé equation can be transformed to a Painlevé III equation.

- Generalized Meixner polynomials with weights $w_k = \frac{(\gamma)_k a^k}{k!(\beta)_k}$ ($a > 0$, $\beta > 0$, $\gamma > 0$) were for $\beta = 1$ investigated in [2] and for general $\beta > 0$ in [22]. The special case $\beta = \gamma$ gives the Charlier polynomials. In [22, Theorem 3.1] it was shown that the recurrence coefficients are given by $a_n^2 = na - (\gamma-1)u_n$, $b_n = n + \gamma - \beta + a - (\gamma-1)v_n/a$, where $(u_n, v_n)_{n\in\mathbb{N}}$ satisfy the system

$$(u_n + v_n)(u_{n+1} + v_n) = \frac{\gamma-1}{a^2} v_n(v_n - a)\left(v_n - a\frac{\gamma-\beta}{\gamma-1}\right),$$
$$(u_n + v_n)(u_n + v_{n-1}) = \frac{u_n}{u_n - \frac{an}{\gamma-1}}(u_n + a)\left(u_n + a\frac{\gamma-\beta}{\gamma-1}\right),$$

with initial conditions $u_0 = 0$ and

$$v_0 = \frac{a}{\gamma-1}\left(\gamma - \beta + a - \frac{\gamma a M(\gamma+1, \beta+1, a)}{\beta M(\gamma, \beta, a)}\right),$$

where $M(a, b, z)$ is the confluent hypergeometric function. This system of non-linear recurrence relations is a limiting case of the asymmetric discrete Painlevé IV equation related to d-P$\big(E_6^{(1)}/A_2^{(1)}\big)$ in [17, Section 8.1.15]. In [10] and [2] it was shown that v_n, as a function of a, satisfies a Painlevé equation. See also [6] for a more direct approach. If we define a function $y_n(a)$ by

$$v_n = \frac{a(ay_n' - (1+\beta-2)y_n^2 + (n+1-a+\beta-2\gamma)y_n - n)}{2(\gamma-1)(y_n-1)y_n},$$

then y_n satisfies Painlevé V

$$y_n'' = \left(\frac{1}{2y_n} + \frac{1}{y_n-1}\right)(y_n')^2 - \frac{y_n'}{a} + \frac{(y_n-1)^2}{a^2}\left(Ay_n + \frac{B}{y_n}\right) + \frac{Cy_n}{a} + \frac{Dy_n(y_n+1)}{y_n-1},$$

with

$$A = \frac{(\beta-1)^2}{2}, \qquad B = -\frac{n^2}{2}, \qquad C = n - \beta + 2\gamma, \qquad D = -\frac{1}{2}.$$

When $\gamma = -N$ is a negative integer one deals with generalized Krawtchouk polynomials, which were investigated in [1].

All these families of discrete orthogonal polynomials are orthogonal on the integers $\mathbb{N} = \{0, 1, 2, \ldots\}$. They were studied by Dominici and Marcellán in [9] who were investigating discrete semi-classical orthogonal polynomials of class one, which also includes the Hahn polynomials (which are orthogonal on a finite set $\{0, 1, 2. \ldots, N\}$). They gave limit relations between these and other families of orthogonal polynomials. The generalized Charlier polynomials and the generalized Meixner polynomials can also be made orthogonal on the shifted lattice $\mathbb{N} + \beta - 1$ if $\beta < 2$, and the corresponding recurrence coefficients satisfy the same Painlevé equations, but with a different initial value for b_0. The more general setting is to consider the generalized Charlier and Meixner polynomials as orthogonal polynomials on the union of $\mathbb{N}$ and $\mathbb{N} + \beta - 1$. See [22] for more details. This general setting on the bi-lattice gives solutions of the Painlevé equations depending on a seed function (the moment m_0) that consists of a linear combination of two solutions of the Bessel equation or the Kummer equation.

There is one case of discrete orthogonal polynomials that has not been considered in much detail and which also has recurrence coefficients that satisfy discrete and continuous Painlevé equations. Take the weights

$$w_k = \frac{(\alpha)_k(\beta)_k}{(\gamma)_k k!}c^k, \qquad \alpha, \beta, \gamma > 0, \qquad 0 < c < 1, \tag{1.2}$$

which corresponds to case 7 in [9]. The initial moment of this weight is

$$m_0 = \sum_{k=0}^{\infty} \frac{(\alpha)_k(\beta)_k}{(\gamma)_k k!} c^k = {}_2F_1(\alpha,\beta;\gamma;c)$$

involving the Gauss hypergeometric function, and all the other moments are

$$m_n = \sum_{k=0}^{\infty} k^n \frac{(\alpha)_k(\beta)_k}{(\gamma)_k k!} c^k = \left(c\frac{\mathrm{d}}{\mathrm{d}c}\right)^n m_0.$$

So therefore one may expect that the recurrence coefficients of the corresponding orthogonal polynomials satisfy a Painlevé VI equation, because Painlevé VI has special function solutions in terms of hypergeometric functions, see [5, Section 7.5]. There are also special function solutions for discrete Painlevé equations, and for a review we refer to [21]. In this paper we will find a system of two first order recurrence relations (see Theorem 3.1) which allows us to deduce some asymptotic behavior as $n \to \infty$ (Section 6). In Section 5 we make the connection with the σ-form of the sixth Painlevé equation (see Theorem 5.1). Note that Dominici already obtained non-linear recurrence relations for the recurrence coefficients in [8, Theorem 4] which he calls the Laguerre–Freud equations. These are however of higher order than two and neither they are identified as discrete Painlevé equations, nor is a connection made with Painlevé VI.

There are also some examples of continuous weights for which the recurrence coefficients of the orthogonal polynomials are related to Painlevé VI. Dai and Zhang [7] and Lyu and Chen [19] considered a generalization of the Jacobi weight function on $[0,1]$,

$$w(x,t) = x^\alpha(1-x)^\beta|x-t|^\gamma, \qquad x \in [0,1],$$

where t is a real parameter, and found that Painlevé VI is the relevant equation for the recurrence coefficients as a function of t. Chen and Zhang [3] investigated another modification of the Jacobi weight on $[0,1]$,

$$w(x,t) = x^\alpha(1-x)^\beta\big(A + B\Theta(x-t)\big), \qquad x \in [0,1],$$

where Θ is the Heaviside function, and showed that Painlevé VI is appearing for the recurrence coefficients of the corresponding orthogonal polynomials. In both cases the moments can be expressed in terms of the Gauss hypergeometric function, and it is the special function solution of Painlevé VI which is needed to find the recurrence coefficients. See also [23, Section 6.2.5] for this connection.

2 Hypergeometric weights

We will investigate the orthogonal polynomials given by the discrete orthogonality relations

$$\sum_{k=0}^{\infty} p_n(k)p_m(k)\frac{(\alpha)_k(\beta)_k}{(\gamma)_k k!} c^k = \delta_{m,n},$$

with $\alpha,\beta,\gamma > 0$ and $0 < c < 1$, and in particular we want to find the recurrence coefficients $(a_n,b_n)_{n\in\mathbb{N}}$ in the three term recurrence relation

$$xp_n(x) = a_{n+1}p_{n+1}(x) + b_np_n(x) + a_np_{n-1}(x). \tag{2.1}$$

The weights $w_k = w(k)$ can be given as the values at the integers $k \in \mathbb{N}$ of the function

$$w(x) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)}\frac{\Gamma(\alpha+x)\Gamma(\beta+x)}{\Gamma(\gamma+x)\Gamma(x+1)}c^x. \tag{2.2}$$

Observe that for $\alpha = \gamma$ one finds the weights for the Meixner polynomials, and for $c \to 0$ and $\alpha \to \infty$, in such a way that $\alpha c \to a > 0$, one finds the generalized Meixner weight, which in turn for $\beta = \gamma$ gives the Charlier weight and for $a \to 0$ and $\beta \to \infty$ in such a way that $\beta a \to \hat{a} > 0$, gives the generalized Charlier weight. We will use the theory of ladder operators for discrete orthogonal polynomials [16], [15, Section 6.3]. This uses a discrete potential

$$u(x) = -\frac{w(x) - w(x-1)}{w(x)} = -1 + \frac{(\gamma + x - 1)x}{c(\alpha + x - 1)(\beta + x - 1)},$$

which is rational, with simple poles at $x = -\alpha$ and $x = -\beta$. Define

$$A_n(x) = a_n \sum_{k=0}^{\infty} p_n(k) p_n(k-1) \frac{u(x+1) - u(k)}{x + 1 - k} w_k,$$

and

$$B_n(x) = a_n \sum_{k=0}^{\infty} p_n(k) p_{n-1}(k-1) \frac{u(x+1) - u(k)}{x + 1 - k} w_k,$$

then one has the structure relation

$$\Delta p_n(x) = A_n(x) p_{n-1}(x) - B_n(x) p_n(x), \tag{2.3}$$

where $\Delta p_n(x) = p_n(x+1) - p_n(x)$ is the forward difference of $p_n(x)$. Some straightforward calculus shows that

$$\frac{u(x+1) - u(k)}{x + 1 - k} = \frac{c_k}{x + \alpha} + \frac{d_k}{x + \beta},$$

for certain sequences (c_k, d_k), so that

$$\frac{A_n(x)}{a_n} = \frac{u_n}{x + \alpha} + \frac{v_n}{x + \beta}, \qquad B_n(x) = \frac{r_n}{x + \alpha} + \frac{s_n}{x + \beta},$$

where (u_n, v_n) and (r_n, s_n) are sequences depending on α, β, γ, c. The compatibility between (2.1) and (2.3) gives the relations

$$B_{n+1}(x) + B_n(x) = \frac{x - b_n}{a_n} A_n(x) - u(x+1) + \sum_{j=0}^{n} \frac{A_j(x)}{a_j}$$

and

$$a_{n+1} A_{n+1}(x) - a_n^2 \frac{A_{n-1}(x)}{a_{n-1}} = (x - b_n) B_{n+1}(x) - (x - b_n + 1) B_n(x) + 1.$$

In our case this gives

$$\begin{aligned} &\frac{r_n}{x+\alpha} + \frac{s_n}{x+\beta} + \frac{r_{n+1}}{x+\alpha} + \frac{s_{n+1}}{x+\beta} \\ &\quad = (x - b_n)\left(\frac{u_n}{x+\alpha} + \frac{v_n}{x+\beta}\right) + 1 - \frac{(\gamma + x)(x+1)}{c(\alpha + x)(\beta + x)} + \sum_{j=0}^{n}\left(\frac{u_j}{x+\alpha} + \frac{v_j}{x+\beta}\right), \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} &a_{n+1}^2\left(\frac{u_{n+1}}{x+\alpha} + \frac{v_{n+1}}{x+\beta}\right) - a_n^2\left(\frac{u_{n-1}}{x+\alpha} + \frac{v_{n-1}}{x+\alpha}\right) \\ &\quad = (x - b_n)\left(\frac{r_{n+1}}{x+\alpha} + \frac{s_{n+1}}{x+\beta}\right) - (x - b_n + 1)\left(\frac{r_n}{x+\alpha} + \frac{s_n}{x+\beta}\right) + 1. \end{aligned} \tag{2.5}$$

Our goal is to determine the unknown sequences a_n, b_n, u_n, v_n, r_n, s_n from the compatibility relations (2.4)–(2.5).

Proposition 2.1. *For $\alpha \neq \beta$ the sequences u_n, v_n, r_n, s_n are given by*

$$(\alpha-\beta)u_n = 2n+1-\frac{1-c}{c}b_n+\frac{\alpha+\beta-\gamma-1}{c}+(n+1-\beta)\frac{1-c}{c}, \tag{2.6}$$

$$(\beta-\alpha)v_n = 2n+1-\frac{1-c}{c}b_n+\frac{\alpha+\beta-\gamma-1}{c}+(n+1-\alpha)\frac{1-c}{c}, \tag{2.7}$$

and

$$(\alpha-\beta)r_n = \frac{n(n-1)}{2}-\frac{1-c}{c}a_n^2+\beta n+\sum_{k=0}^{n-1}b_k, \tag{2.8}$$

$$(\beta-\alpha)s_n = \frac{n(n-1)}{2}-\frac{1-c}{c}a_n^2+\alpha n+\sum_{k=0}^{n-1}b_k. \tag{2.9}$$

Observe the symmetry $v_n(\alpha,\beta,\gamma,c)=u_n(\beta,\alpha,\gamma,c)$ and $s_n(\alpha,\beta,\gamma,c)=r_n(\beta,\alpha,\gamma,c)$, which holds because a_n and b_n are invariant if we interchange α and β (interchanging α and β leaves the weights w_k unchanged).

Proof. The identity (2.4) gives three equations by looking at what happens for $x\to\infty$, $-\alpha$, and $-\beta$. If we let $x\to\infty$ in (2.4) then

$$u_n+v_n=\frac{1-c}{c}, \tag{2.10}$$

the residue at $x=-\alpha$ gives

$$r_n+r_{n+1}=-u_n(\alpha+b_n)-\frac{(\gamma-\alpha)(1-\alpha)}{c(\beta-\alpha)}+\sum_{j=0}^{n}u_j, \tag{2.11}$$

and the residue at $x=-\beta$ gives

$$s_n+s_{n+1}=-v_n(\beta+b_n)-\frac{(\gamma-\beta)(1-\beta)}{c(\alpha-\beta)}+\sum_{j=0}^{n}v_j. \tag{2.12}$$

In a similar way we get three equations from (2.5): first let $x\to\infty$ to find

$$(r_{k+1}+s_{k+1})-(r_k+s_k)=-1,$$

which after summation (and using $r_0+s_0=0$, which follows because $a_0=0$) gives

$$r_n+s_n=-n. \tag{2.13}$$

The residue at $x=-\alpha$ for (2.5) gives

$$a_{n+1}^2u_{n+1}-a_n^2u_{n-1}=-r_{n+1}(\alpha+b_n)+r_n(\alpha+b_n-1), \tag{2.14}$$

and the residue at $x=-\beta$ gives

$$a_{n+1}^2v_{n+1}-a_n^2v_{n-1}=-s_{n+1}(\beta+b_n)+s_n(\beta+b_n-1). \tag{2.15}$$

Adding (2.11) and (2.12) while using (2.10) and (2.13) gives

$$\alpha u_n+\beta v_n=2n+1-\frac{1-c}{c}b_n+\frac{\alpha+\beta-\gamma-1}{c}+(n+1)\frac{1-c}{c}. \tag{2.16}$$

Now we can solve the linear system (2.10) and (2.16) for (u_n, v_n) and this gives the required expressions (2.6)–(2.7). In a similar way, we add (2.14) and (2.15), which together with (2.10) and (2.13) gives

$$\left(a_{n+1}^2 - a_n^2\right)\frac{1-c}{c} = b_n + n - (\alpha r_{n+1} + \beta s_{n+1}) + (\alpha r_n + \beta s_n).$$

Summing this then gives (taking into account that $a_0 = 0$)

$$\alpha r_n + \beta s_n = \frac{n(n-1)}{2} - \frac{1-c}{c} a_n^2 + \sum_{k=0}^{n-1} b_k. \tag{2.17}$$

Now we can solve the linear system (2.13) and (2.17) for the unknowns (r_n, s_n) to find the expressions (2.8)–(2.9). ■

Corollary 2.2. *The recurrence coefficients* $\left(a_n^2, b_n\right)_{n\in\mathbb{N}}$ *are given in terms of* $(u_n, r_n)_{n\in\mathbb{N}}$ *by*

$$b_n = \frac{n+\alpha-\gamma+(n+\beta)c}{1-c} - (\alpha-\beta)\frac{c}{1-c}u_n, \tag{2.18}$$

$$a_n^2 = \frac{n(n+\alpha+\beta-\gamma-1)c}{(1-c)^2} - (\alpha-\beta)\frac{c}{1-c}\left(\frac{c}{1-c}\sum_{j=0}^{n-1} u_j + r_n\right). \tag{2.19}$$

Proof. The formula (2.18) follows immediately from (2.6). Summing (2.18) gives

$$\beta n + \sum_{k=0}^{n-1} b_k = \frac{n(n-1)}{2}\frac{1+c}{1-c} + \frac{n(\alpha+\beta-\gamma)}{1-c} - (\alpha-\beta)\frac{c}{1-c}\sum_{j=0}^{n-1} u_j,$$

and if we use this in (2.8), then we find

$$\frac{1-c}{c}a_n^2 = \frac{n(n+\alpha+\beta-\gamma-1)}{1-c} - (\alpha-\beta)\left(\frac{c}{1-c}\sum_{j=0}^{n-1} u_j + r_n\right),$$

from which (2.19) follows immediately. ■

Note that for $\alpha = \gamma$ the weights w_k become Meixner weights, and if we use the recurrence coefficients in (1.1), then one finds that $u_n = 0 = r_n$ for all $n \in \mathbb{N}$. The restriction that $\alpha \neq \beta$ in Proposition 2.1 is not needed but is an artifact of our choice of taking u_n and r_n as the basic sequences. In fact, when $\alpha = \beta$ the discrete potential u has a double pole, resulting in a double pole for A_n and B_n as well. In the next section we will use new variables x_n and y_n for which the case $\alpha = \beta$ needs no separate analysis.

3 New variables

Recall that

$$\frac{A_n}{a_n} = \frac{u_n}{x+\alpha} + \frac{v_n}{x+\beta} = \frac{(u_n+v_n)x + \beta u_n + \alpha v_n}{(x+\alpha)(x+\beta)},$$

and

$$B_n = \frac{r_n}{x+\alpha} + \frac{s_n}{x+\beta} = \frac{(r_n+s_n)x + \beta r_n + \alpha s_n}{(x+\alpha)(x+\beta)}.$$

We already found that $u_n + v_n = \frac{1-c}{c}$, see (2.10), and $r_n + s_n = -n$, see (2.13), so we now take new variables

$$x_n = \frac{c}{1-c}(\beta u_n + \alpha v_n), \qquad y_n = \beta r_n + \alpha s_n,$$

and thus use

$$\frac{A_n}{a_n} = \frac{1-c}{c}\frac{x+x_n}{(x+\alpha)(x+\beta)}, \qquad B_n = \frac{-nx+y_n}{(x+\alpha)(x+\beta)}. \tag{3.1}$$

The advantage of using the unknowns (x_n, y_n) is that these are symmetric in α and β: they remain unchanged if one interchanges α and β. We are also going to use one more compatibility relation between the A_n and B_n. We already know

$$B_{n+1}(x) + B_n(x) = (x-b_n)\frac{A_n(x)}{a_n} - u(x+1) + \sum_{k=0}^{n}\frac{A_k(x)}{a_k}, \tag{3.2}$$

and

$$a_{n+1}^2\frac{A_{n+1}(x)}{a_{n+1}} - a_n^2\frac{A_{n-1}(x)}{a_{n-1}} = (x-b_n)\big(B_{n+1}(x) - B_n(x)\big) + 1 - B_n(x). \tag{3.3}$$

Multiply (3.3) by A_n/a_n, then

$$a_{n+1}^2\frac{A_{n+1}A_n}{a_{n+1}a_n} - a_n^2\frac{A_nA_{n-1}}{a_na_{n-1}} = (x-b_n)\frac{A_n}{a_n}(B_{n+1}-B_n) + (1-B_n)\frac{A_n}{a_n}.$$

Replace $(x-b_n)A_n/a_n$ by using (3.2), then

$$a_{n+1}^2\frac{A_{n+1}A_n}{a_{n+1}a_n} - a_n^2\frac{A_nA_{n-1}}{a_na_{n-1}}$$
$$= B_{n+1}^2 - B_n^2 + (B_{n+1}-B_n)\left(u(x+1) - \sum_{k=0}^{n}\frac{A_k}{a_k}\right) + (1-B_n)\frac{A_n}{a_n}.$$

Summing from 0 to $n-1$, taking into account that $A_{-1}=0$ and $B_0=0$, gives

$$a_n^2\frac{A_nA_{n-1}}{a_na_{n-1}} = B_n^2 + B_nu(x+1) - \sum_{k=0}^{n-1}(B_{k+1}-B_k)\sum_{j=0}^{k}\frac{A_j}{a_j} + \sum_{k=0}^{n-1}\frac{A_k}{a_k} - \sum_{k=0}^{n-1}B_k\frac{A_k}{a_k}.$$

Use summation by parts (for $f_0 = 0$)

$$\sum_{k=0}^{n-1}(g_{k+1}-g_k)f_k = g_nf_{n-1} - \sum_{k=1}^{n-1}g_k(f_k - f_{k-1}) \tag{3.4}$$

to find our third compatibility relation

$$a_n^2\frac{A_n(x)A_{n-1}(x)}{a_na_{n-1}} = B_n^2(x) + u(x+1)B_n(x) + \big(1 - B_n(x)\big)\sum_{k=0}^{n-1}\frac{A_k(x)}{a_k}. \tag{3.5}$$

Compare this with [23, equation (4.8)] for the ladder operators corresponding to the differential operator.

Now let us find some relations for the unknown sequences $\left(a_n^2, b_n\right)_n$ and $(x_n, y_n)_n$. If we use (3.1) in (3.2) and multiply everything by $(x+\alpha)(x+\beta)$, then we find a quadratic expression in x for which the quadratic term vanishes, so that we only have a linear term in x and a constant term. The coefficient of x gives the identity

$$b_n = x_n + \frac{n + (n+\alpha+\beta)c - \gamma}{1-c}, \tag{3.6}$$

which corresponds to (2.18) in Corollary 2.2. The constant term gives

$$y_{n+1} + y_n = -\frac{1-c}{c} b_n x_n + \alpha\beta - \frac{\gamma}{c} + \frac{1-c}{c}\sum_{k=0}^{n} x_k. \tag{3.7}$$

Next we use (3.1) in (3.3) and multiply everything by $(x+\alpha)(x+\beta)$. Again this gives a quadratic equation in x in which the quadratic term vanishes. The linear term gives

$$\frac{1-c}{c}\left(a_{n+1}^2 - a_n^2\right) = y_{n+1} - y_n + b_n + \alpha + \beta + n. \tag{3.8}$$

Summing from 0 to $n-1$ and using (3.6) gives

$$\frac{1-c}{c} a_n^2 = y_n + \sum_{k=0}^{n-1} x_k + \frac{n(n+\alpha+\beta-\gamma-1)}{1-c}, \tag{3.9}$$

which corresponds to (2.19) in Corollary 2.2. The constant term gives

$$\frac{1-c}{c}\left(a_{n+1}^2 x_{n+1} - a_n^2 x_{n-1}\right) = -b_n(y_{n+1} - y_n) + \alpha\beta - y_n. \tag{3.10}$$

Multiply this by $\frac{1-c}{c} x_n$ and use (3.7) to eliminate $b_n x_n$, then

$$\begin{aligned} &\frac{(1-c)^2}{c^2}\left(a_{n+1}^2 x_{n+1} x_n - a_n^2 x_n x_{n-1}\right) \\ &\qquad = y_{n+1}^2 - y_n^2 - \left(\alpha\beta - \frac{\gamma}{c}\right)(y_{n+1} - y_n) - (y_{n+1} - y_n)\frac{1-c}{c}\sum_{k=0}^{n} x_k + \frac{1-c}{c} x_n(\alpha\beta - y_n). \end{aligned}$$

Summing and using summation by parts (3.4) then gives

$$\frac{(1-c)^2}{c^2} a_n^2 x_n x_{n-1} = y_n\left(y_n - \alpha\beta + \frac{\gamma}{c}\right) - (y_n - \alpha\beta)\frac{1-c}{c}\sum_{k=0}^{n-1} x_k. \tag{3.11}$$

Finally, we use (3.1) in (3.5) and multiply everything by $(x+\alpha)^2(x+\beta)^2$. This gives a cubic equation in x in which the cubic term vanishes. The quadratic term gives (3.9) again. The linear term gives

$$\begin{aligned} \frac{(1-c)^2}{c^2} a_n^2 (x_n + x_{n-1}) &= -y_n\left(n\frac{1+c}{c} + \alpha + \beta - \frac{\gamma+1}{c}\right) + \frac{(\alpha\beta-\gamma)n}{c} \\ &\quad + (\alpha+\beta+n)\frac{1-c}{c}\sum_{k=0}^{n-1} x_k, \end{aligned} \tag{3.12}$$

and the constant term gives (3.11) again.

So now we have equations (3.6) and (3.9) to express the recurrence coefficients a_n^2 and b_n in terms of the sequences $(x_n, y_n)_n$. Furthermore we will use (3.7), (3.11) and (3.12) to find a system of recurrence relations for $(x_n, y_n)_n$. Note that these three equations contain the sum $\sum_{k=0}^{n-1} x_k$ which we will eliminate from these equations so that we are left with a system of two first order equations for $(x_n, y_n)_n$.

Theorem 3.1. *The sequences* (x_n, y_n) *can be computed recursively using*

$$\begin{aligned}\big(y_n - \alpha\beta + (\alpha+\beta+n)x_n - x_n^2\big)\big(y_{n+1} - \alpha\beta + (\alpha+\beta+n+1)x_n - x_n^2\big)\\ = \frac{1}{c}(x_n-1)(x_n-\alpha)(x_n-\beta)(x_n-\gamma),\end{aligned} \tag{3.13}$$

and

$$\begin{aligned}&(x_n+Y_n)(x_{n-1}+Y_n)\\ &= \frac{(y_n+n\alpha)(y_n+n\beta)\big(y_n+n\gamma-(\gamma-\alpha)(\gamma-\beta)\big)\big(y_n+n-(1-\alpha)(1-\beta)\big)}{\Big(y_n(2n+\alpha+\beta-\gamma-1)+n\big((n+\alpha+\beta)(n+\alpha+\beta-\gamma-1)-\alpha\beta+\gamma\big)\Big)^2},\end{aligned} \tag{3.14}$$

where

$$Y_n = \frac{y_n^2 + y_n\big(n(n+\alpha+\beta-\gamma-1)-\alpha\beta+\gamma\big) - \alpha\beta n(n+\alpha+\beta-\gamma-1)}{y_n(2n+\alpha+\beta-\gamma-1)+n\big((n+\alpha+\beta)(n+\alpha+\beta-\gamma-1)-\alpha\beta+\gamma\big)}.$$

The initial values are given by

$$y_0 = 0, \qquad x_0 = \frac{c\alpha\beta}{\gamma}\frac{{}_2F_1(\alpha+1,\beta+1;\gamma+1;c)}{{}_2F_1(\alpha,\beta;\gamma;c)} - \frac{(\alpha+\beta)c-\gamma}{1-c}.$$

Proof. Multiply (3.7) by y_n to find

$$y_{n+1}y_n + y_n^2 = -\frac{1-c}{c}b_nx_ny_n + \alpha\beta y_n - \frac{\gamma}{c}y_n + y_n\frac{1-c}{c}S_n + \frac{1-c}{c}x_ny_n, \tag{3.15}$$

where from now on we write

$$S_n = \sum_{k=0}^{n-1} x_k.$$

Multiply (3.12) by x_n and use (3.6) to find

$$\frac{1-c}{c}a_n^2\big(x_n^2 + x_nx_{n-1}\big) = -b_nx_ny_n + x_n^2y_n + \frac{x_ny_n}{1-c} + \frac{(\alpha\beta-\gamma)nx_n}{1-c} + (\alpha+\beta+n)x_nS_n.$$

Use (3.11) to remove $a_n^2x_nx_{n-1}$ and then (3.9) to remove the remaining a_n^2 to find

$$\begin{aligned}x_n^2\left(S_n + \frac{n(n+\alpha+\beta-\gamma-1)}{1-c}\right) + \frac{c}{1-c}y_n\left(y_n - \alpha\beta + \frac{\gamma}{c}\right) - (y_n-\alpha\beta)S_n\\ = -b_nx_ny_n + \frac{x_ny_n}{1-c} + \frac{(\alpha\beta-\gamma)nx_n}{1-c} + (\alpha+\beta+n)x_nS_n.\end{aligned} \tag{3.16}$$

Eliminate $b_nx_ny_n$ from (3.15) and (3.16) to find

$$\begin{aligned}y_{n+1}y_n - \frac{n(n+\alpha+\beta-\gamma-1)}{c}x_n^2 + x_ny_n + \frac{(\alpha\beta-\gamma)nx_n}{c}\\ = \big(\alpha\beta - (\alpha+\beta+n)x_n + x_n^2\big)\frac{1-c}{c}S_n.\end{aligned} \tag{3.17}$$

Now use (3.7) and (3.6) to find

$$\frac{1-c}{c}S_n = y_{n+1} + y_n + \frac{1-c}{c}x_n\left(x_n + \frac{n+(n+\alpha+\beta)c-\gamma}{1-c}\right) - \alpha\beta + \frac{\gamma}{c} - \frac{1-c}{c}x_n$$

and use this to replace the sum S_n in (3.17). This gives an expression containing x_n and the terms $y_{n+1}+y_n$ and y_ny_{n+1}. Some calculus shows that it can be factored as in (3.13).

Next, use (3.9) to replace a_n^2 in (3.11) and (3.12). Then one can eliminate the sum $S_n = \sum_{k=0}^{n-1} x_k$ from both equations and find

$$\begin{aligned} x_nx_{n-1}\big(y_n(2n+\alpha+\beta-\gamma-1) + n\big((n+\alpha+\beta)(n+\alpha+\beta-\gamma-1)-\alpha\beta+\gamma\big)\big) \\ + (x_n+x_{n-1})\big(y_n^2 + y_n\big(n(n+\alpha+\beta-\gamma-1)-\alpha\beta+\gamma\big) - \alpha\beta n(n+\alpha+\beta-\gamma-1)\big) \\ = y_n^2(-n+\gamma+1) + y_n(2\alpha\beta n+\alpha\gamma+\beta\gamma-\alpha\beta\gamma-\alpha\beta) - \alpha\beta(\alpha\beta-\gamma)n. \end{aligned}$$

This is an equation containing y_n and the sum x_n+x_{n-1} and product x_nx_{n-1}. Some (lengthy) calculus shows it can be factored as in (3.14).

The initial values are $a_0^2 = 0$ and $b_0 = m_1/m_0$, where $m_0 = {}_2F_1(\alpha,\beta;\gamma;c)$ and $m_1 = \frac{c\alpha\beta}{\gamma}{}_2F_1(\alpha+1,\beta+1;\gamma+1;c)$, which gives

$$y_0 = 0, \qquad x_0 = \frac{m_1}{m_0} - \frac{(\alpha+\beta)c-\gamma}{1-c}. \qquad \blacksquare$$

The system (3.13)–(3.14) is a system of two first order recurrence equations for x_n, y_n and is a discrete Painlevé equation, similar to d-P$\big(E_6^{(1)}/A_2^{(1)}\big)$ in [17, Section 8.1.15] or [13, E_6^δ on p. 296], except for a quadratic term x_n^2 on the left of (3.13) and the rational term Y_n on the left of (3.14).

4 The Toda lattice

If we put $c = c_0e^t$ then the weight (2.2) is an exponential modification of the weight with $c = c_0$ for $t = 0$, and this deformation (with deformation parameter t) corresponds to a Toda flow. The recurrence coefficients $a_n^2(t)$ and $b_n(t)$, as functions of the deformation parameter t, then satisfy the Toda equations

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}a_n^2 &= a_n^2(b_n-b_{n-1}), \qquad n\ge 1, \\ \frac{\mathrm{d}}{\mathrm{d}t}b_n &= a_{n+1}^2 - a_n^2, \qquad n \ge 0, \end{aligned}$$

see, e.g., [15, Section 2.8] or [23, Section 3.2.2]. In the variable c, these Toda equations become

$$c\frac{\mathrm{d}}{\mathrm{d}c}a_n^2 = a_n^2(b_n-b_{n-1}), \qquad n\ge 1, \tag{4.1}$$

$$c\frac{\mathrm{d}}{\mathrm{d}c}b_n = a_{n+1}^2 - a_n^2, \qquad n \ge 0. \tag{4.2}$$

For the sequences x_n and y_n we then have:

Proposition 4.1. *The derivatives of x_n and y_n with respect to the parameter c are given by*

$$x_n' = b_n' - \frac{2n+\alpha+\beta-\gamma}{(1-c)^2}, \tag{4.3}$$

and

$$y_n' = -\frac{1+c}{c^2}a_n^2 + \frac{1-c}{c}\big(a_n^2\big)', \tag{4.4}$$

where $'$ denotes derivation with respect to c. Furthermore the Toda equations for (x_n, y_n) are

$$(1-c)x_n' = y_{n+1} - y_n + x_n, \tag{4.5}$$

$$(1-c)y_n' = \frac{(1-c)^2}{c^2}a_n^2(x_n-x_{n-1}), \qquad n\ge 0. \tag{4.6}$$

Proof. If we take the derivative with respect to c in (3.6) then we find (4.3). In a similar way we take the derivative in (3.9) and (4.3) to find

$$y_n' = -\frac{a_n^2}{c^2} + \frac{1-c}{c}\left(a_n^2\right)' - \sum_{k=0}^{n-1} b_k'.$$

Now use the Toda equation (4.2) to find

$$\sum_{k=0}^{n-1} b_k' = \frac{1}{c}\sum_{k=0}^{n-1}\left(a_{k+1}^2 - a_k^2\right) = \frac{a_n^2}{c},$$

where we used $a_0^2 = 0$. This gives (4.4). If we use (4.2), then (4.3) becomes

$$x_n' = \frac{a_{n+1}^2 - a_n^2}{c} - \frac{2n+\alpha+\beta-\gamma}{(1-c)^2},$$

which after using (3.9) gives (4.5). If we use (4.1), then (4.4) becomes

$$y_n' = -\frac{1+c}{c^2}a_n^2 + \frac{1-c}{c^2}a_n^2(b_n - b_{n-1}),$$

and (3.6) then gives (4.6). ■

5 Painlevé VI

By combining the Toda equations (4.5)–(4.6) with the discrete Painlevé equations (3.13)–(3.14) one can in principle find a differential equation for x_n and y_n as a function of the variable c, which after a suitable transformation can be reduced to Painlevé VI. This approach is rather cumbersome and we were able to work it out by using computer algebra. However, we will present here another approach which gives an easier differential equation for $S_n = \sum_{k=0}^{n-1} x_k$.

Theorem 5.1. *If we put* $\sigma_n = (c-1)S_n + Kc + L$, *with*

$$K = \alpha\beta - \frac{1}{4}(\alpha+\beta+n)^2,$$
$$L = \frac{1}{4}\left((\alpha+\beta+\gamma+1)n + \alpha^2 + \beta^2 - (\alpha+\beta)(\gamma+1) + 2\gamma\right),$$

then σ_n *satisfies the Painlevé VI* σ*-equation*

$$\sigma_n'\left[c(c-1)\sigma_n''\right]^2 + \left[\sigma_n'\left(2\sigma_n - (2c-1)\sigma_n'\right) + d_1d_2d_3d_4\right]^2 = \left(\sigma_n' + d_1^2\right)\left(\sigma_n' + d_2^2\right)\left(\sigma_n' + d_3^2\right)\left(\sigma_n' + d_4^2\right), \tag{5.1}$$

with parameters

$$d_1 = \frac{n+\alpha-\beta}{2}, \qquad d_2 = \frac{-n+\alpha-\beta}{2}, \qquad d_3 = \frac{n+\alpha+\beta-2}{2}, \qquad d_4 = \frac{n+\alpha+\beta-2\gamma}{2}.$$

Proof. Again we consider S_n (and σ_n) as a function of the variable c and derivatives are with respect to c. Summing (4.5) and using $y_0 = 0$ shows that

$$(1-c)S_n' = y_n + S_n,$$

and hence

$$y_n = [(1-c)S_n]', \qquad y_n' = [(1-c)S_n]''. \tag{5.2}$$

Subtracting (3.12) and (4.6) gives

$$2\frac{(1-c)^2}{c^2}a_n^2 x_n = (1-c)y_n' - y_n\left(n\frac{1+c}{c} + \alpha + \beta - \frac{\gamma+1}{c}\right) + \frac{(\alpha\beta-\gamma)n}{c} + (\alpha+\beta+n)\frac{1-c}{c}S_n, \tag{5.3}$$

while adding (3.12) and (4.6) gives

$$2\frac{(1-c)^2}{c^2}a_n^2 x_{n-1} = -(1-c)y_n' - y_n\left(n\frac{1+c}{c} + \alpha + \beta - \frac{\gamma+1}{c}\right) + \frac{(\alpha\beta-\gamma)n}{c} + (\alpha+\beta+n)\frac{1-c}{c}S_n.$$

If we multiply both expressions, then

$$4\frac{(1-c)^4}{c^4}a_n^4 x_n x_{n-1} = \left[-y_n\left(n\frac{1+c}{c} + \alpha + \beta - \frac{\gamma+1}{c}\right) + \frac{(\alpha\beta-\gamma)n}{c} + (\alpha+\beta+n)\frac{1-c}{c}S_n\right]^2 - \left[(1-c)y_n'\right]^2. \tag{5.4}$$

Recall that (3.9) gives

$$\frac{(1-c)^2}{c^2}a_n^2 = \frac{1-c}{c}(y_n+S_n) + \frac{n(n+\alpha+\beta-\gamma-1)}{c},$$

hence combining this with (3.11) gives

$$4\frac{(1-c)^4}{c^4}a_n^4 x_n x_{n-1} = 4\left(\frac{1-c}{c}(y_n+S_n) + \frac{n(n+\alpha+\beta-\gamma-1)}{c}\right) \times \left(y_n\left(y_n - \alpha\beta + \frac{\gamma}{c}\right) - (y_n-\alpha\beta)\frac{1-c}{c}S_n\right). \tag{5.5}$$

Clearly (5.4)–(5.5) gives the equation

$$\begin{aligned}&\left[-y_n\left(n\frac{1+c}{c} + \alpha + \beta - \frac{\gamma+1}{c}\right) + \frac{(\alpha\beta-\gamma)n}{c} + (\alpha+\beta+n)\frac{1-c}{c}S_n\right]^2 - \left[(1-c)y_n'\right]^2 \\ &\quad = 4\left(\frac{1-c}{c}(y_n+S_n) + \frac{n(n+\alpha+\beta-\gamma-1)}{c}\right) \\ &\quad \times \left(y_n\left(y_n - \alpha\beta + \frac{\gamma}{c}\right) - (y_n-\alpha\beta)\frac{1-c}{c}S_n\right),\end{aligned}$$

and if we replace y_n and y_n' by (5.2), then this is a non-linear second order differential equation for S_n, or better for $\hat{\sigma}_n = (c-1)S_n$:

$$\begin{aligned}&[c(1-c)\hat{\sigma}_n'']^2 + 4\big((c-1)\hat{\sigma}_n' - \hat{\sigma}_n + n(n+\alpha+\beta-\gamma-1)\big) \\ &\quad \times \big(\hat{\sigma}_n'(c\hat{\sigma}_n' + \alpha\beta c - \gamma) - (\hat{\sigma}_n' + \alpha\beta)\hat{\sigma}_n\big) \\ &\quad = \big[\hat{\sigma}_n'\big(n + (\alpha+\beta+n)c - \gamma - 1\big) + (\alpha\beta-\gamma)n - (\alpha+\beta+n)\hat{\sigma}_n\big]^2.\end{aligned}$$

If we now put $\sigma_n = \hat{\sigma}_n + Kc + L$, then a lengthy but straightforward computation gives the required Painlevé VI σ-equation. ■

The Painlevé σ-equations are given as equations E_I–E_{VI} in [20] and σ-equations σPII–σPVI are given in [12, equation (8.15) in Section 8.1 or equation (8.29) in Section 8.2]. They are equivalent to the six Painlevé equations P_I–P_{VI} in the sense that there is a one-to-one correspondence between solutions of the Painlevé equations and the corresponding σ-equations. Clearly our solution σ_n is a special function solution which is expressed in terms of a Wronskian containing hypergeometric functions.

If σ_n is known, then also S_n is known, and then from (5.2) it follows that $y_n = [(1-c)S_n]'$. Using (3.9) we find

$$\frac{1-c}{c} a_n^2 = y_n + S_n + \frac{n(n+\alpha+\beta-\gamma-1)}{1-c},$$

so that the recurrence coefficient a_n^2 is in terms of S_n and S_n'. For x_n one can use (5.3) to find that it is in terms of S_n, S_n' and S_n''. Then finally (3.6) shows that b_n is also in terms of S_n, S_n' and S_n''. Hence S_n and its first two derivatives are enough to find the quantities of interest for these discrete orthogonal polynomials. Furthermore S_n'' is in terms of S_n and S_n' because of the σ-equation (5.1).

Remark. Special function solutions of Painlevé VI are generated by a seed function that comes from a Riccati equation; see, e.g., [5, Section 7.5] or [23, Section 6.2.5]. We can show that x_n indeed satisfies a Riccati equation and in particular that x_0 is the seed function. Differentiate (4.5) with respect to c to find

$$(1-c)x_n'' = y_{n+1}' - y_n' + 2x_n'.$$

Replace y_{n+1}' and y_n' by using the Toda equation (4.6), then

$$(1-c)^2 x_n'' = \frac{(1-c)^2}{c^2}\big(a_{n+1}^2(x_{n+1}-x_n) - a_n^2(x_n - x_{n-1})\big) + 2(1-c)x_n'. \tag{5.6}$$

Combining (3.8) with (3.6) and (4.5) gives

$$\frac{(1-c)^2}{c^2}\big(a_{n+1}^2 - a_n^2\big) = \frac{(1-c)^2}{c} x_n' + \frac{2n+\alpha+\beta-\gamma}{c}.$$

Multiply this by x_n and add this to (5.6) to find

$$\begin{aligned}(1-c)^2 x_n'' &+ \frac{(1-c)^2}{c} x_n x_n' + \frac{2n+\alpha+\beta-\gamma}{c} x_n \\ &= \frac{(1-c)^2}{c^2}\big(a_{n+1}^2 x_{n+1} + a_n^2 x_{n-1} - 2a_n^2 x_n\big) + 2(1-c)x_n'.\end{aligned} \tag{5.7}$$

Multiply (3.10) by $\frac{1-c}{c}$ and add this to (5.7) to find

$$\begin{aligned}(1-c)^2 x_n'' &+ \frac{(1-c)^2}{c} x_n x_n' + \frac{2n+\alpha+\beta-\gamma}{c} x_n + \frac{1-c}{c}(y_n - \alpha\beta) \\ &+ \frac{1-c}{c}\left(x_n + \frac{n+(n+\alpha+\beta)c-\gamma}{1-c}\right)(y_{n+1}-y_n) \\ &= 2\frac{(1-c)^2}{c^2} a_n^2 (x_{n-1} - x_n) + 2(1-c)x_n'.\end{aligned}$$

Use (4.5) and (4.6) to replace $y_{n+1} - y_n$ and $x_n - x_{n-1}$ and collect terms to find

$$\begin{aligned}c(1-c)x_n'' &+ 2(1-c)x_n x_n' + \big(n + (n+\alpha+\beta-2)c - \gamma\big)x_n' - x_n^2 + (n+\alpha+\beta)x_n - \alpha\beta \\ &= -y_n - 2cy_n'.\end{aligned} \tag{5.8}$$

This equation contains only the functions x_n'', x_n', x_n and the functions y_n', y_n. Observe that the left hand side can be written as

$$\big(c(1-c)x_n' + (1-c)x_n^2 + \big(n+(n+\alpha+\beta)c-\gamma-1\big)x_n - \alpha\beta c\big)'$$

so that (5.8) is in fact a Riccati equation for x_n if y_n is given. Recall that $y_0 = 0$, therefore we have the Riccati equation for x_0

$$c(1-c)x_0' + (1-c)x_0^2 + \big((\alpha+\beta)c-\gamma-1\big)x_0 - \alpha\beta c = \text{constant}.$$

One can verify this, using the fact that

$$x_0(c) = \frac{c\alpha\beta}{\gamma}\frac{{}_2F_1(\alpha+1,\beta+1;\gamma+1;c)}{{}_2F_1(\alpha,\beta;\gamma;c)} - \frac{(\alpha+\beta)c-\gamma}{1-c},$$

and it turns out that the constant is $-\gamma$. This Riccati equation for x_0 gives a seed function for all the special function solutions x_n.

6 Asymptotic behavior

As we mentioned before, the case $\alpha = \gamma$ (or $\beta = \gamma$) gives the Meixner polynomials for which the recurrence coefficients are known, see (1.1). If we compare this with (3.6) and (3.9), then it follows that for this special case

$$x_n = \gamma, \qquad y_n = -n\gamma.$$

The sequence $(x_n)_n$ is a constant sequence and the constant is a zero of the right hand side of (3.13).

In Fig. 1 we computed the $(x_n, y_n)_n$ for $\alpha = 3/2$, $\beta = 3$, $\gamma = 1/3$ and $c = 1/2$ for the weights w_k in (1.2) on the integers $\mathbb{N} = \{0,1,2,3,\ldots\}$ by using the recurrence (3.13)–(3.14). We used a precision of `Digits:=50` in Maple, because for `Digits:=10` the resulting values for x_n, y_n were wrong when $n \geq 40$. The precision `Digits:=20` gives the same plots and the computed values only go wrong for $n \geq 80$. The initial values are

$$y_0 = 0, \qquad x_0 = \frac{m_1}{m_0} - \frac{(\alpha+\beta)c-\gamma}{1-c},$$

where

$$m_0 = {}_2F_1(\alpha,\beta;\gamma;c), \qquad m_1 = \frac{c\alpha\beta}{\gamma}{}_2F_1(\alpha+1,\beta+1;\gamma+1;c).$$

The calculations seem to suggest that x_n converges to γ and that y_n decreases linearly for large n.

We conjecture that for this initial value for x_0

$$\lim_{n\to\infty} x_n = \gamma$$

and then (3.13) implies that

$$\lim_{n\to\infty} (y_n + n\gamma) = (\gamma-\alpha)(\gamma-\beta).$$

Note that the latter is a zero of the right hand side of (3.14). This asymptotic behavior seems to be confirmed by the numerical results. We believe that this is the only initial value which

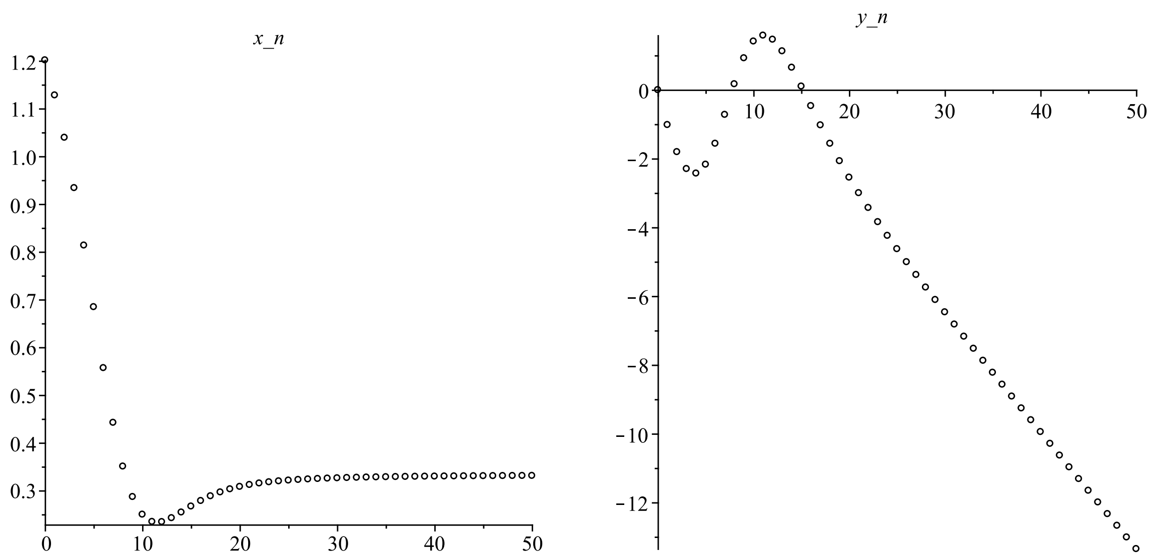

Figure 1. The sequences x_n (left) and y_n (right) for $(\alpha,\beta,\gamma,c) = \left(\frac{3}{2}, 3, \frac{1}{3}, \frac{1}{2}\right)$.

gives this asymptotic behavior. The calculations are very sensitive of the initial value: a slight change of x_0 gives a more erratic behavior of the (x_n, y_n) for large n.

We can also use the weights (1.2) on the shifted lattice $\mathbb{N}+1-\gamma$ since $w(-\gamma) = 0$ for the function w in (2.2), in a similar way as was done in [22, Sections 2.2 and 3.2]. The same recurrence relations still hold for the corresponding recurrence coefficients $\hat{a}_n, \hat{b}_n$. These recurrence coefficients are related to the recurrence coefficients of the weights on $\mathbb{N}$ by

$$\hat{a}_n^2(\alpha,\beta,\gamma,c) = a_n^2(\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma, c),$$
$$\hat{b}_n(\alpha,\beta,\gamma,c) = b_n(\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma, c) + 1 - \gamma.$$

We conjecture that in this case

$$\lim_{n\to\infty} \hat{x}_n = 1$$

and then (3.13) gives

$$\lim_{n\to\infty} (\hat{y}_n + n) = (1-\alpha)(1-\beta).$$

The corresponding initial values are

$$\hat{y}_0 = 0, \qquad \hat{x}_0 = x_0(\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma, c) + \gamma - 1.$$

Again we believe this is the only initial value for which this asymptotic behavior holds. An interesting question is to find out which initial values give an asymptotic behavior of the form

$$\lim_{n\to\infty} x_n = \alpha, \qquad \text{or} \qquad \lim_{n\to\infty} x_n = \beta,$$

which are the other two zeros of the right hand side of (3.13).

7 Concluding remarks

The reason why we considered the hypergeometric weights (1.2) in this paper is twofold. On one hand they generalize various other discrete weights that were already analyzed in the literature (Charlier, Meixner, generalized Charlier, generalized Meixner, generalized Krawtchouk), building up from explicit rational expressions for the recurrence coefficients to second order nonlinear recurrence and differential equations (Painlevé III and Painlevé V). On the other hand, it was already known that Painlevé VI has special function solutions in terms of Wronskians with hypergeometric functions [23, Section 6.2.5], and such Wronskians appear naturally in formulas for the recurrence coefficients a_n^2 and b_n for orthogonal polynomials:

$$a_n^2 = \frac{\Delta_{n+1}\Delta_{n-1}}{\Delta_n^2}, \qquad b_n = \frac{\Delta_{n+1}^*}{\Delta_{n+1}} - \frac{\Delta_n^*}{\Delta_n},$$

where $\Delta_n = \det(m_{i+j})_{i,j=0}^{n-1}$ is the Hankel determinant and Δ_n^* is obtained from Δ_n by replacing the last column $(m_{n-1}, m_n, \dots, m_{2n-2})^{\mathrm{T}}$ by $(m_n, m_{n+1}, \dots, m_{2n-1})^{\mathrm{T}}$ and $(m_n)_{n\in\mathbb{N}}$ are the moments

$$m_n = \sum_{k=0}^{\infty} k^n w_k.$$

The moment m_0 is a Gauss hypergeometric series and all the other moments can be obtained from them by

$$m_n = \left(c\frac{\mathrm{d}}{\mathrm{d}c}\right)^n m_0,$$

so that Δ_n and Δ_n^* are Wronskians. The challenge was to find the discrete Painlevé equations (Theorem 3.1) and the continuous Painlevé equation (Theorem 5.1) for the recurrence coefficients of the orthogonal polynomials with these hypergeometric weights, using only standard properties of orthogonal polynomials and a number of suitable transformations. The system of discrete Painlevé equations (3.13)–(3.14) seems to be new but closely related to d-P$\big(E_6^{(1)}/A_2^{(1)}\big)$, and the Painlevé equation (5.1) is the σ-form of the Painlevé VI equation.

Note that one can write the weights in (1.2) as

$$w_k = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(\alpha+k)\Gamma(\beta+k)}{\Gamma(\gamma+k)\Gamma(k+1)} c^k := w(k)$$

and that $w(-1) = 0$, which gives a boundary condition for the weights on the lattice $\mathbb{N} = \{0, 1, 2, 3, \dots\}$. One also has $w(-\gamma) = 0$ so that one can also use these weights on the shifted lattice $\mathbb{N} + 1 - \gamma = \{1-\gamma, 2-\gamma, 3-\gamma, \dots\}$, as was done for generalized Charlier and Meixner polynomials in [22]. The recurrence coefficients will satisfy the same discrete Painlevé equations but with a different initial value because m_0 and m_1 are different hypergeometric series: one has $a_0^2 = 0$ and $b_0 = m_1/m_0$ with

$$\begin{aligned} m_0 &= \frac{\Gamma(\gamma)}{\Gamma(\alpha)\gamma(\beta)} \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+k+1-\gamma)\Gamma(\beta+k+1-\gamma)}{\Gamma(k+1)\Gamma(k+2-\gamma)} c^{k+1-\gamma} \\ &= \frac{\Gamma(\gamma)\Gamma(\alpha+1-\gamma)\Gamma(\beta+1-\gamma)}{\Gamma(2-\gamma)\Gamma(\alpha)\Gamma(\beta)} c^{1-\gamma} {}_2F_1(\alpha+1-\gamma, \beta+1-\gamma; 2-\gamma; c), \end{aligned}$$

and $m_1 = c\frac{\mathrm{d}}{\mathrm{d}c} m_0$. The same Painlevé VI σ-equation will hold, but the solution will be a different special function solution coming from a different hypergeometric seed function. The general case is obtained by taking a combination of both lattices, giving a one parameter family of special function solutions of Painlevé VI.

Note that nonlinear recurrence relations for the recurrence coefficients of these orthogonal polynomials were found by Dominici in [8, Theorem 4], but these were of higher order and were not identified as discrete Painlevé equations. Our version (3.13)–(3.14) has the advantage that one can predict the asymptotic behavior of a_n^2 and b_n (or x_n and y_n) as $n \to \infty$ from them, and in Section 6 we conjectured this asymptotic behavior when the weights are on the lattice $\mathbb{N}$ and on the shifted lattice $\mathbb{N} + 1 - \gamma$.

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