Symmetry, Integrability and Geometry: Methods and Applications SIGMA **14** (2018), 134, 18 pages

A Product on Double Cosets of B_∞

Pablo GONZALEZ PAGOTTO

Institut Fourier, Université Grenoble Alpes, Grenoble, France
E-mail: *pablo.gonzalez-pagotto@univ-grenoble-alpes.fr*
URL: https://www-fourier.ujf-grenoble.fr/~gonzpabl/

Received May 28, 2018, in final form December 14, 2018; Published online December 27, 2018
https://doi.org/10.3842/SIGMA.2018.134

Abstract. For some infinite-dimensional groups G and suitable subgroups K there exists a monoid structure on the set $K\backslash G/K$ of double cosets of G with respect to K. In this paper we show that the group B_∞, of the braids with finitely many crossings on infinitely many strands, admits such a structure.

Key words: Braid group; double cosets; Burau representation

2010 Mathematics Subject Classification: 20F36; 20M99; 20C99

1 Introduction

1.1 Motivation

For some infinite-dimensional groups G and suitable subgroups K there exists a monoid structure on the set $K\backslash G/K$ of double cosets of G with respect to K. This can be seen, for example, for the group S_∞ of the finitely supported permutations of $\mathbb{N}$, for infinite-dimensional classical Lie groups, for groups of automorphisms of measure spaces and for $\mathrm{Aut}(F_\infty)$, a direct limit of the groups of automorphisms of the free groups F_n.

The study of these structures was pioneered by R.S. Ismagilov, followed by G.I. Ol'shanski, who used them in the representation theory of infinite-dimensional classical Lie groups [15, 16, 17, 18, 19, 21]. More recently there is the work of Yu.A. Neretin for the infinite tri-symmetric group and $\mathrm{Aut}(F_\infty)$ [8, 11, 13, 14].

In this paper we show that the group B_∞, of the finite braids on infinitely many strands, admits such a structure. Furthermore, we show how this multiplicative structure is related to similar constructions in $\mathrm{Aut}(F_\infty)$ and $\mathrm{GL}(\infty)$. We also define a one-parameter generalization of the usual monoid structure on the set of double cosets of $\mathrm{GL}(\infty)$ (see [9, 10]) and show that the Burau representation provides a functor between the categories of double cosets of B_∞ and $\mathrm{GL}(\infty)$.

1.2 The infinite braid group and double cosets

The *Artin braid group* on n strings B_n [2, 5, 6] has the presentation with $n-1$ generators $\sigma_1, \sigma_2, \dots, \sigma_{n-1}$ and the so-called *braid relations*:

$$\sigma_i\sigma_j = \sigma_j\sigma_i, \qquad |i-j| \geq 2, \qquad i,j \in \{1,\dots,n-1\},$$

and

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \qquad 1 \leq i \leq n-2.$$

The generators σ_i are called *elementary braids*. For each n, consider the monomorphism $i_n\colon B_n \to B_{n+1}$ sending the k-th elementary braid of B_n to the k-th elementary braid of B_{n+1}.

Geometrically this operation corresponds to adding a new string to the right of the braid, without creating any new crossings, as in the picture below:

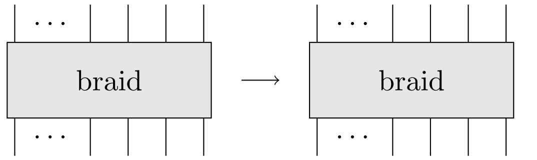


Figure 1. The monomorphism i_n.

The direct limit of this sequence of groups, with respect to the homomorphisms i_n, is the *infinite braid group*

$$B_\infty = \varinjlim B_n,$$

consisting of braids with countably many strings and finitely many crossings. This group has the presentation

$$B_\infty = \left\langle \sigma_i, i \in \mathbb{N} \;\middle|\; \begin{array}{c} \sigma_i\sigma_j = \sigma_j\sigma_i, |i-j| \geq 2 \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \end{array} \right\rangle.$$

For each non-negative integer α, let $B_\infty[\alpha]$ be the subgroup of B_∞ given by

$$B_\infty[\alpha] = \langle \sigma_j \,|\, j > \alpha \rangle.$$

Definition 1.1. Let G be a group, $g \in G$ and K and L be subgroups of G. The double coset on G containing g with respect to the pair (K, L) is the set KgL. Denote by $K\backslash G/L$ the set of double cosets on G with respect to the pair (K, L).

1.3 The Burau representation of B_∞

The Burau representation [3, 4] is the homomorphism $\eta_n \colon B_n \to \mathrm{GL}\big(n, \mathbb{Z}[t, t^{-1}]\big)$ given by

$$\eta_n(\sigma_i) = \begin{pmatrix} 1_{i-1} & & \\ & \begin{pmatrix} 1-t & t \\ 1 & 0 \end{pmatrix} & \\ & & 1_{n-i-1} \end{pmatrix}.$$

Denote $\mathrm{GL}\big(n, \mathbb{Z}[t, t^{-1}]\big)$ by $\mathrm{GL}(n)$ and consider the homomorphisms $j_n \colon \mathrm{GL}(n) \to \mathrm{GL}(n+1)$ given by

$$j_n(T) = \begin{pmatrix} T & 0 \\ 0 & 1 \end{pmatrix}.$$

The group $\mathrm{GL}(\infty)$ is the direct limit of $\mathrm{GL}(n)$ with respect to the homomorphisms j_n and consists of infinite matrices that differ from the identity matrix only in finitely many entries. Due to the commutativity of the diagram

$$\begin{array}{ccc} B_n & \xrightarrow{\eta_n} & \mathrm{GL}(n) \\ {\scriptstyle i_n}\downarrow & & \downarrow{\scriptstyle j_n} \\ B_{n+1} & \xrightarrow[\eta_{n+1}]{} & \mathrm{GL}(n+1) \end{array}$$

we can construct a representation $\eta\colon B_\infty \to \mathrm{GL}(\infty)$ of B_∞ by taking the limit of the representations η_n. More precisely, η is given by the following formulas:

$$\eta(\sigma_i) = \begin{pmatrix} 1_{i-1} & & \\ & \begin{pmatrix} 1-t & t \\ 1 & 0 \end{pmatrix} & \\ & & 1_\infty \end{pmatrix}.$$

With this representation in mind, we will define an operation on double cosets of $\mathrm{GL}(\infty)$ such that the Burau representation will be functorial between the categories of double cosets.

1.4 Main results

Consider the double cosets on B_∞ with respect to the subgroups $B_\infty[\alpha]$. Given double cosets $\mathfrak{p} \in B_\infty[\alpha]\backslash B_\infty/B_\infty[\beta]$ and $\mathfrak{q} \in B_\infty[\beta]\backslash B_\infty/B_\infty[\gamma]$, we are going to define an element $\mathfrak{p} \circ \mathfrak{q} \in B_\infty[\alpha]\backslash B_\infty/B_\infty[\gamma]$. To this purpose, we first introduce the following:

Definition 1.2. For integers $\beta \geq 0$ and $n > 0$, denote by $\tau_i^{(n)}$ the braid

$$\tau_i^{(n)} = \sigma_{n+\beta+i}\sigma_{n+\beta+i-1}\cdots\sigma_{\beta+i+1}.$$

Further we define the element $\theta_n[\beta] \in B_\infty[\beta]$ as

$$\theta_n[\beta] = \tau_0^{(n)}\tau_1^{(n)}\cdots\tau_{n-1}^{(n)}.$$

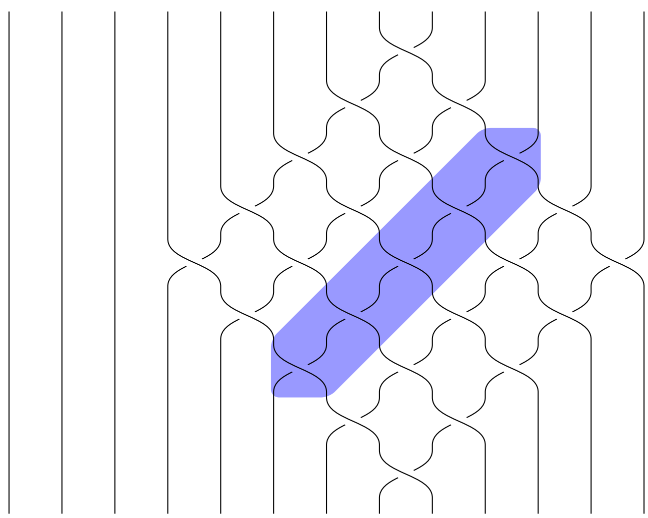


Figure 2. The element $\theta_5[3]$. The highlighted region corresponds to $\tau_2^{(5)}$.

Finally, the definition of the product of the double cosets is as follows:

Definition 1.3. Let $\mathfrak{p} \in B_\infty[\alpha]\backslash B_\infty/B_\infty[\beta]$ and $\mathfrak{q} \in B_\infty[\beta]\backslash B_\infty/B_\infty[\gamma]$ be double cosets. Consider $p \in \mathfrak{p}$ and $q \in \mathfrak{q}$ representatives of these double cosets. Then we define their product as

$$\mathfrak{p} \circ \mathfrak{q} = B_\infty[\alpha]p\theta_n[\beta]qB_\infty[\gamma],$$

for sufficiently large n.

Remark 1.4. The introduction of the element $\theta_k[\beta]$ is essential for our construction of a product on the set of double cosets of B_∞. In fact, for $\mathfrak{p} \in B_\infty[\alpha]\backslash B_\infty/B_\infty[\beta]$ and $\mathfrak{q} \in B_\infty[\beta]\backslash B_\infty/B_\infty[\gamma]$, let $p \in \mathfrak{p}$ and $q \in \mathfrak{q}$ be representatives of these double cosets. The "naive" product $B_\infty[\alpha]pqB_\infty[\gamma]$ does not always coincide for all choices of p and q. For instance σ_2 and $\sigma_3\sigma_2$ are representatives of the same double coset in $B_\infty[2]\backslash B_\infty/B_\infty[2]$. But σ_2^2 and $\sigma_3\sigma_2\sigma_3\sigma_2$ represent distinct cosets. In order to see this, we consider the permutation associated to each braid. For the braid σ_2^2 it is the identity and for the braid $\sigma_3\sigma_2\sigma_3\sigma_2$ it is (432). Since no braid in $B_\infty[2]$ permutes the point 2, we see that these are in fact distinct double cosets.

However, if we introduce an intermediary braid $\theta_k[\beta]$ that "forces apart" the braids p and q, the double coset $B_\infty[\alpha]p\theta_k[\beta]qB_\infty[\gamma]$ becomes independent of k for k large enough and its limit does not depend on the choice of the representatives for $\mathfrak{p}$ and $\mathfrak{q}$.

Theorem 1.5. *The operation defined above does not depend on the choice of the representatives of the double cosets for n large enough. Moreover, it is associative.*

As a consequence we have that $(B_\infty[\alpha]\backslash B_\infty/B_\infty[\alpha], \circ)$ is a monoid, for each non-negative integer α.

Remark 1.6. We will show that there exists some $n_0(\alpha,\gamma,p,q)$ such that, for all $n \geq n_0$, $B_\infty[\alpha]p\theta_n[\beta]qB_\infty[\gamma] = B_\infty[\alpha]p\theta_{n_0}[\beta]qB_\infty[\gamma]$. More precisely, $n_0 = \max\{\operatorname{supp} p, \operatorname{supp} q, \alpha, \gamma\}+1$, where supp is the support of a braid, defined in Definition 2.1.

For a group G and a subgroup $H \subset G$, we say that $g, g' \in G$ are conjugate with respect to H if there exists $h \in H$ such that $g' = hgh^{-1}$. Denote by $G//H$ the set of conjugacy classes with respect to the subgroup H. There is a natural one-to-one correspondence between the sets $H\backslash(G^n \times H)/H$ and $G^n//H$ (here H is the image of the subgroup $H \subset G$ by the appropriate diagonal map). In fact, it is easy to see that the function from $G^n \times H$ to G^n given by

$$(g_1, g_2, \dots, g_n, h) \mapsto \left(g_1h^{-1}, g_2h^{-1}, \dots, g_nh^{-1}\right),$$

induces a bijection between the sets $H\backslash(G^n \times H)/H$ and $G^n//H$.

Using the correspondence above, we can define a monoid structure on the set $B_\infty//B_\infty[\alpha]$. In fact, we have an one-to-one correspondence between the sets $B_\infty//B_\infty[\alpha]$ and $B_\infty[\alpha]\backslash(B_\infty \times B_\infty[\alpha])/B_\infty[\alpha]$, the later being a submonoid of $B_\infty[\alpha]\backslash(B_\infty \times B_\infty)/B_\infty[\alpha]$.

Furthermore, as a consequence of the existence of a solution for the conjugacy problem for the braid groups and the fact that the injections i_n do not merge conjugacy classes (see [7]), we have

Proposition 1.7. *The conjugacy problem for B_∞ has a solution.*

Notice that combining the observations above with Proposition 1.7, it is possible to devise an algorithm to determine when two elements of $B_\infty \times B_\infty$ belong to the same class in $B_\infty[0]\backslash B_\infty \times B_\infty/B_\infty[0]$.

Now, let $v = \left(1, t, t^2, \dots\right)$ and $u = (1, 1, 1, 1, \dots)$ and denote by x^{T} the transpose of the vector x. Consider the subgroup of $\mathrm{GL}(\infty)$ given by

$$G[n] = \left\{ \begin{pmatrix} 1_n & \\ & X \end{pmatrix}; \, X \in \mathrm{GL}(\infty),\, v^{\mathrm{T}}X = v^{\mathrm{T}},\, Xu = u \right\}.$$

It is easy to see that the image of $B_\infty[n]$ by the Burau representation is contained in $G[n]$.

Definition 1.8. Consider the matrix

$$\Theta_j[k] = \begin{pmatrix} \mathbf{1}_k & 0 & 0 & 0 \\ 0 & V_j & t^j 1_j & 0 \\ 0 & 1_j & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1}_\infty \end{pmatrix},$$

where

$$V_j = (1-t)\begin{pmatrix} 1 & t & \cdots & t^{j-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & t & \cdots & t^{j-1} \end{pmatrix}.$$

Let $p \in \mathfrak{p}$ and $q \in \mathfrak{q}$ be representatives of the double cosets $\mathfrak{p} \in G[n]\backslash \mathrm{GL}(\infty)/G[k]$ and $\mathfrak{q} \in G[k]\backslash \mathrm{GL}(\infty)/G[m]$. Then we define their product as

$$\mathfrak{p} \star_t \mathfrak{q} = G[n]p\Theta_j[k]qG[m],$$

for sufficiently large j.

Theorem 1.9. *The operation defined above does not depend on the choice of representatives of double cosets for j large enough. Moreover, it is associative.*

Remark 1.10. In particular, there exists an integer $j_0(n,m,p,q)$ such that $G[n]p\Theta_j[k]qG[m] = G[n]p\Theta_{j_0}[k]qG[m]$ for all $j \geq j_0$. We can make j_0 more precise. In fact let $N \in \mathbb{N}$ be such that p and q can be written as diagonal block matrices $\left(\begin{smallmatrix} A & 0 \\ 0 & 1_\infty \end{smallmatrix}\right)$, where A is a square matrix of dimension $k+N$. Then $j_0 = \max\{m, n, k+N\}$.

Remark 1.11. The operation $\star_t$ generalizes the usual multiplication defined on the double cosets of $\mathrm{GL}(\infty)$ in the sense that setting the parameter $t=1$ we recover the usual multiplication.

Let G be a group and $K[*] = \{K[s]; s \in \mathbb{N}\}$ a family of subgroups of G. We say that there is a well-defined operation on the double cosets of G with relation to the family $K[*]$ when there exists a family of morphisms

$$\mu = \{\mu_{rst}\colon K[r]\backslash G/K[s] \times K[s]\backslash G/K[t] \to K[r]\backslash G/K[t]; r,s,t \in \mathbb{N}\}$$

satisfying

$$\mu_{rtu}(\mu_{rst} \times 1_{K[t]\backslash G/K[u]}) = \mu_{rsu}(1_{K[r]\backslash G/K[s]} \times \mu_{stu})$$

for all $r,s,t,u \in \mathbb{N}$ and, if $\mathbf{e} \in K[r]\backslash G/K[r]$ denotes the class of the unit element of G, then for all $\alpha \in K[t]\backslash G/K[r]$ and all $\beta \in K[r]\backslash G/K[t]$

$$\mu_{trr}(\alpha, \mathbf{e}) = \alpha \qquad \text{and} \qquad \mu_{rrt}(\mathbf{e}, \beta) = \beta.$$

In this case, consider the category $\mathcal{K}(G,K)$ of double cosets, where the objects are nonnegative integers and the morphisms are given by $\mathrm{Hom}(r,s) = K[s]\backslash G/K[r]$. Then,

Proposition 1.12. *The Burau representation $\eta\colon B_\infty \to \mathrm{GL}(\infty)$ induces a functor between the categories $\mathcal{K}(B_\infty, B_\infty[*])$ and $\mathcal{K}(\mathrm{GL}(\infty), G[*])$.*

When G is the bisymmetric group (the group that consists of pairs (g,h) of permutations of $\mathbb{N}$ such that gh^{-1} is a finite permutation) and K is its diagonal subgroup, we get a category called the train category of the pair (G,K). This category admits a transparent combinatorial description and encodes information about the representations of the bisymmetric group (see [12, 20]).

2 Proofs of main results

2.1 Proof of Theorem 1.5

Before proceeding, we introduce the notion of *support*, which will be needed later.

Definition 2.1. Let p be a braid in B_∞. The *support* of p is

$$\operatorname{supp} p = \min\{j \in \mathbb{N};\, p \in \langle \sigma_1, \dots, \sigma_j\rangle\}.$$

Notice that the decomposition of p into the product of elementary braids does not contain any element of $B_\infty[\operatorname{supp} p]$, hence p commutes with every element of $B_\infty[1+\operatorname{supp} p]$. Also, we can identify p with an element of $B_{1+\operatorname{supp} p}$. We define $\operatorname{supp} 1 = 0$.

Consider double cosets

$$\mathfrak{p} \in B_\infty[\alpha]\backslash B_\infty/B_\infty[\beta] \qquad \text{and} \qquad \mathfrak{q} \in B_\infty[\beta]\backslash B_\infty/B_\infty[\gamma],$$

and let $p \in \mathfrak{p}$ and $q \in \mathfrak{q}$ be their respective representatives. Setting $\mathfrak{r}_j = B_\infty[\alpha]p\theta_j[\beta]qB_\infty[\gamma]$, we have a sequence of double cosets in $B_\infty[\alpha]\backslash B_\infty/B_\infty[\gamma]$.

Proposition 2.2. *The sequence $(\mathfrak{r}_j)_{j\geq 1}$ defined above is eventually constant.*

Proof. We are going to give a proof in several steps:

Step 1. Given $m > 0$ we have $\tau_i^{(m+1)} = \sigma_{m+\beta+1+i}\tau_i^{(m)}$ for all $0 \leq i \leq m-1$. In fact, we have the equality

$$\tau_i^{(m+1)} = \sigma_{m+1+\beta+i}\sigma_{m+\beta+i}\sigma_{m+\beta+i-1}\cdots\sigma_{\beta+i+1} = \sigma_{m+1+\beta+i}\tau_i^{(m)}.$$

Step 2. For all $j \leq i$ we have $\sigma_{m+\beta+i+2}\tau_j^{(m)} = \tau_j^{(m)}\sigma_{m+\beta+i+2}$. Indeed, since $\operatorname{supp}\tau_j^{(m)} = m+\beta+j$ and $\sigma_{m+\beta+i+2} \in B_\infty[1+m+\beta+j]$, we find that $\sigma_{m+\beta+i+2}$ commutes with $\tau_j^{(m)}$.

Step 3. Define $u = (\sigma_{m+\beta+1}\sigma_{m+\beta+2}\cdots\sigma_{2m+\beta})^{-1}$ and $\ell^{-1} = \tau_m^{(m+1)}$. Then $\theta_m[\beta] = u\theta_{m+1}[\beta]\ell$. In fact, we have

$$\begin{aligned}
u\theta_{m+1}[\beta]\ell &= u\big(\tau_0^{(m+1)}\cdots\tau_m^{(m+1)}\big)\ell = u\tau_0^{(m+1)}\cdots\tau_{m-1}^{(m+1)}\\
&= u\big(\sigma_{m+\beta+1}\tau_0^{(m)}\big)\big(\sigma_{m+\beta+2}\tau_1^{(m)}\big)\cdots\big(\sigma_{2m+\beta}\tau_{m-1}^{(m)}\big)\\
&= \sigma_{2m+\beta}^{-1}\sigma_{2m+\beta-1}^{-1}\cdots\sigma_{m+\beta+2}^{-1}\tau_0^{(m)}\big(\sigma_{m+\beta+2}\tau_1^{(m)}\big)\cdots\big(\sigma_{2m+\beta}\tau_{m-1}^{(m)}\big)\\
&= \sigma_{2m+\beta}^{-1}\sigma_{2m+\beta-1}^{-1}\cdots\sigma_{m+\beta+2}^{-1}\sigma_{m+\beta+2}\tau_0^{(m)}\tau_1^{(m)}\cdots\big(\sigma_{2m+\beta}\tau_{m-1}^{(m)}\big)\\
&= \sigma_{2m+\beta}^{-1}\sigma_{2m+\beta-1}^{-1}\cdots\sigma_{m+\beta+3}^{-1}\tau_0^{(m)}\tau_1^{(m)}\big(\sigma_{m+\beta+3}\tau_2^{(m)}\big)\cdots\big(\sigma_{2m+\beta}\tau_{m-1}^{(m)}\big)\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&= \tau_0^{(m)}\tau_1^{(m)}\tau_2^{(m)}\cdots\tau_{m-1}^{(m)} = \theta_m[\beta].
\end{aligned}$$

Step 4. Let $M = \max\{\operatorname{supp} p, \operatorname{supp} q, \alpha, \gamma\} + 1$. We show that for all $m \geq M$, we have $\mathfrak{r}_m = \mathfrak{r}_{m+1}$ and hence that $\mathfrak{r}_m = \mathfrak{r}_M$ for all $m \geq M$. Let u and ℓ be like in step 3. Since $u, \ell \in B_\infty[m+\beta]$, it follows that $u \in B_\infty[\alpha]$, $\ell \in B_\infty[\gamma]$ and they commute with p and q. Therefore

$$u(p\theta_{m+1}[\beta]q)\ell = p(u\theta_{m+1}[\beta]\ell)q = p\theta_m[\beta]q.$$

Thus $\mathfrak{r}_m = \mathfrak{r}_{m+1}$. ■

The following technical lemma will be used in the proof of Lemma 2.5, which in turn is used in Proposition 2.7 and more extensively in Theorem 2.8.

Lemma 2.3. *Let* $\{(\upsilon^i_j)_{j=1}^{\ell}\}_{i=1}^{g}$ *be a family of sequences of positive integers such that* $\upsilon^i_{j+k} < \upsilon^{i+n}_j$ *whenever* $k+n>0$ *with* $k,n\in\mathbb{N}$*; in other words, the sequences* $(\upsilon^i_j)_{j\geq 1}$ *are decreasing and the sequences* $(\upsilon^i_j)_{i\geq 1}$ *are increasing. If* $\mu_j = \prod_{k=1}^{\ell}\sigma_{\upsilon^j_k}$ *and* $\lambda_i = \prod_{k=1}^{g}\sigma_{\upsilon^k_i}$*, then* $\mu_1\cdots\mu_g = P = \lambda_1\cdots\lambda_\ell$.

Proof. We prove the lemma by induction on the pair (g,ℓ). The statement is trivial for $g=\ell=1$. Assume it is true for (g,ℓ), we prove it is true for $(g+1,\ell)$ and $(g,\ell+1)$.

For $(g+1,\ell)$, notice that

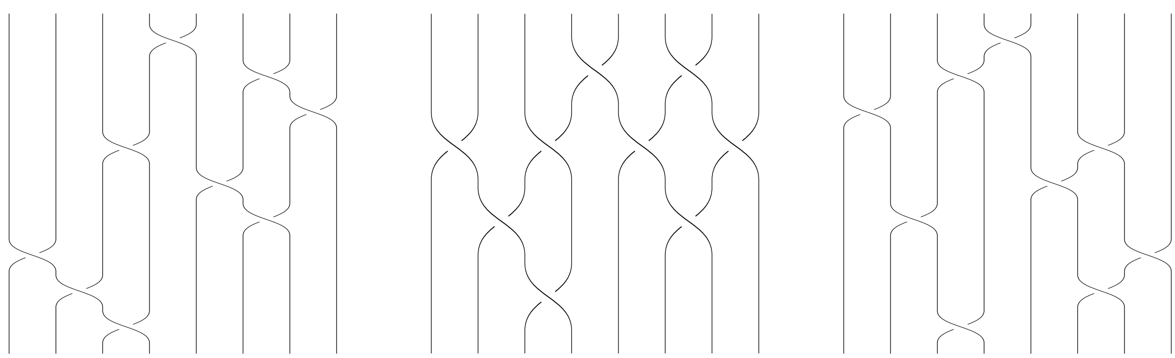
$$\prod_{s=1}^{g+1}\prod_{r=1}^{\ell}\sigma_{\upsilon^s_r} = \left(\prod_{s=1}^{g}\prod_{r=1}^{\ell}\sigma_{\upsilon^s_r}\right)\left(\prod_{r=1}^{\ell}\sigma_{\upsilon^{g+1}_r}\right) = \left(\prod_{r=1}^{\ell}\prod_{s=1}^{g}\sigma_{\upsilon^s_r}\right)\left(\prod_{r=1}^{\ell}\sigma_{\upsilon^{g+1}_r}\right).$$

If $x_r = \prod_{s=1}^{g}\sigma_{\upsilon^s_r}$ we have that $x_r\sigma_{\upsilon^{g+1}_t} = \sigma_{\upsilon^{g+1}_t}x_r$ for $r>t$, this follows from the inequalities $\upsilon^s_r < \upsilon^{g+1}_r < \upsilon^{g+1}_t$ for $s<g+1$. Therefore

$$\begin{aligned}\left(\prod_{r=1}^{\ell}x_r\right)\left(\prod_{r=1}^{\ell}\sigma_{\upsilon^{g+1}_r}\right) &= x_1\cdots x_\ell\sigma_{\upsilon^{g+1}_1}\cdots\sigma_{\upsilon^{g+1}_\ell} = x_1\sigma_{\upsilon^{g+1}_1}x_2\sigma_{\upsilon^{g+1}_2}\cdots x_\ell\sigma_{\upsilon^{g+1}_\ell}\\ &= \prod_{r=1}^{\ell}x_r\sigma_{\upsilon^{g+1}_r} = \prod_{r=1}^{\ell}\prod_{s=1}^{g+1}\sigma_{\upsilon^s_r}.\end{aligned}$$

The proof for the case $(g,\ell+1)$ is analogous. ∎

Example 2.4. Consider the sequences given by $\upsilon^i_j = 3+i-j$, $1\leq i,j\leq 3$. Let μ_i, λ_i, $i=1,2,3$ be as in Lemma 2.3. By the same lemma we have that $\mu_1\mu_2\mu_3 = \lambda_1\lambda_2\lambda_3$. These products are depicted in Fig. 3(a) and (c). Drawing these braids in a more compact form (Fig. 3(b)) the equivalence between the products becomes evident.



(a) $\mu_1\mu_2\mu_3$ (b) P (c) $\lambda_1\lambda_2\lambda_3$

Figure 3. The equality $\mu_1\mu_2\mu_3 = P = \lambda_1\lambda_2\lambda_3$.

It will be useful to write the product P from Lemma 2.3 as a matrix, where the indices increase from right to left and from top to bottom.

$$P = \begin{bmatrix}\upsilon^1_1 & \to & \upsilon^1_\ell\\ \downarrow & & \downarrow\\ \upsilon^g_1 & \to & \upsilon^g_\ell\end{bmatrix}.$$

In this way, $\lambda_1 \cdots \lambda_\ell$ is the column-wise product and $\mu_1 \cdots \mu_g$ is the row-wise product.

Consider, for each positive integer m, the homomorphism $C_m\colon B_\infty \to B_\infty$ given by $C_m(\sigma_j) = \sigma_{m+j}$. Then we have the following lemma.

Lemma 2.5. *Let β and j be nonnegative integers with $j > 1$. If $d \in \langle \sigma_{\beta+1}, \dots, \sigma_{\beta+j-1} \rangle$, then:*

(i) $d\theta_j[\beta] = \theta_j[\beta] C_j(d)$,

(ii) $\theta_j[\beta] d = C_j(d)\theta_j[\beta]$.

Proof. Since C_j is a homomorphism, it is enough to prove both statements of the proposition for the case where $d = \sigma_k$, for some $\beta + 1 \le k \le \beta + j - 1$.

(i) Recall that $\theta_j[\beta] = \tau_0^{(j)} \cdots \tau_{j-1}^{(j)}$. We claim that the following holds:

$$\sigma_{k+i}\tau_i^{(j)} = \tau_i^{(j)}\sigma_{k+i+1}, \qquad 0 \le i \le j-1.$$

Indeed, since σ_{k+i} is a letter of $\tau_i^{(j)}$, but it is different from $\sigma_{j+\beta+i}$, we have

$$\begin{aligned}
\sigma_{k+i}\tau_i^{(j)} &= \sigma_{k+i}(\sigma_{j+\beta+i} \cdots \sigma_{\beta+1+i}) = \sigma_{j+\beta+i} \cdots \sigma_{k+i+2}\sigma_{k+i}\sigma_{k+i+1}\sigma_{k+i}\sigma_{k+i-1} \cdots \sigma_{\beta+1+i} \\
&= \sigma_{j+\beta+i} \cdots \sigma_{k+i+2}\sigma_{k+i+1}\sigma_{k+i}\sigma_{k+i+1}\sigma_{k+i-1} \cdots \sigma_{\beta+1+i} \\
&= \sigma_{j+\beta+i} \cdots \sigma_{\beta+1+i}\sigma_{k+i+1} = \tau_i^{(j)}\sigma_{k+i+1}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\sigma_k\theta_j[\beta] &= \sigma_k\tau_0^{(j)} \cdots \tau_{j-1}^{(j)} = \tau_0^{(j)}\sigma_{k+1}\tau_1^{(j)} \cdots \tau_{j-1}^{(j)} = \cdots = \tau_0^{(j)} \cdots \sigma_{k+j-1}\tau_{j-1}^{(j)} \\
&= \tau_0^{(j)} \cdots \tau_{j-1}^{(j)}\sigma_{k+j} = \theta_j[\beta]\sigma_{k+j}.
\end{aligned}$$

(ii) Let $\upsilon_r^s = j + \beta + s - r$ for r and s positive integers. The family $\{(\upsilon_r^s)\}_{r,s=1}^{j}$ satisfies the hypothesis of Lemma 2.3 and therefore $\mu_1 \cdots \mu_j = \lambda_1 \cdots \lambda_j$, where

$$\mu_i = \sigma_{j+\beta+i-1} \cdots \sigma_{\beta+i} \qquad \text{and} \qquad \lambda_i = \sigma_{j+\beta-i+1} \cdots \sigma_{2j+\beta-i}.$$

Since $\mu_i = \tau_{i-1}^{(j)}$, we see that $\theta_j[\beta] = \lambda_1 \cdots \lambda_j$. As we saw in item (i), we have that

$$\lambda_{j-i}\sigma_{k+i} = \sigma_{k+i+1}\lambda_{j-i}, \qquad 0 \le i \le j-1. \qquad \blacksquare$$

Remark 2.6. The intuition behind Lemma 2.5 is that the element $\theta_j[\beta]$ exchanges braids in the interval between strands $\beta+1$ and $\beta+j$ with braids in the interval between strands $\beta+j+1$ and $\beta+2j$.

Our next step is to prove that the product does not depend on the chosen representatives.

Proposition 2.7. *Let p' and q' be other two representatives of $\mathfrak{p}$ and $\mathfrak{q}$ respectively. Consider the sequence*

$$\mathfrak{r}'_j = B_\infty[\alpha]p'\theta_j[\beta]q'B_\infty[\gamma].$$

Then there exists an integer $N > 0$ such that

$$\mathfrak{r}'_j = \mathfrak{r}_j, \qquad \textit{for all} \quad j \ge N.$$

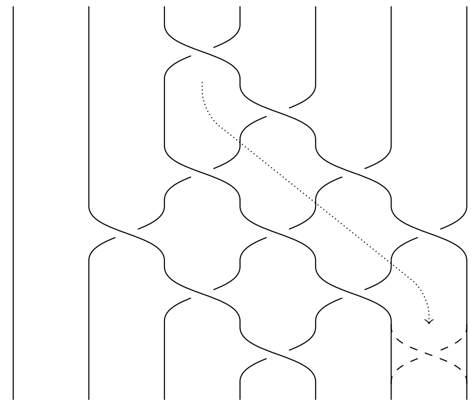


Figure 4. The element $\theta_3[1]$ exchanges the elementary braids σ_3 and σ_6.

Proof. Since p and p' are representatives of the same double coset, there exist $r \in B_\infty[\alpha]$ and $h \in B_\infty[\beta]$ such that $p' = rph$. In a similar way, there exist $k \in B_\infty[\beta]$ and $s \in B_\infty[\gamma]$ such that $q' = kqs$. Therefore,

$$\mathfrak{r}'_j = B_\infty[\alpha]p'\theta_j[\beta]q'B_\infty[\gamma] = B_\infty[\alpha]rph\theta_j[\beta]kqsB_\infty[\gamma] = B_\infty[\alpha]ph\theta_j[\beta]kqB_\infty[\gamma].$$

Consider $N = \max\{\operatorname{supp} p, \operatorname{supp} q, \operatorname{supp} h, \operatorname{supp} k, \alpha, \gamma\} + 1$. Given $j \geq N$, let $\bar{h} = C_j(h^{-1})$ and $\bar{k} = C_j(k^{-1})$. Then $\bar{h}, \bar{k} \in B_\infty[j + \beta]$ and hence $\bar{h} \in B_\infty[\gamma]$ and $\bar{k} \in B_\infty[\alpha]$. Furthermore, $\bar{h}$ commutes with q and k, and $\bar{k}$ commutes with p and h. Now,

$$\begin{aligned} \bar{k}ph\theta_j[\beta]kq\bar{h} &= ph\bar{k}\theta_j[\beta]k\bar{h}q = ph\bar{k}C_j(k)\theta_j[\beta]\bar{h}q \\ &= phC_j(k^{-1})C_j(k)\theta_j[\beta]\bar{h}q = ph\theta_j[\beta]\bar{h}q = p\theta_j[\beta]C_j(h)\bar{h}q = p\theta_j[\beta]q. \end{aligned}$$

■

Therefore, for all pairs $(\mathfrak{p}, \mathfrak{q}) \in B_\infty[\alpha]\backslash B_\infty/B_\infty[\beta] \times B_\infty[\beta]\backslash B_\infty/B_\infty[\gamma]$ we have a well-defined product $\mathfrak{p} \circ \mathfrak{q} \in B_\infty[\alpha]\backslash B_\infty/B_\infty[\gamma]$ given by

$$\mathfrak{p} \circ \mathfrak{q} = B_\infty[\alpha]p\theta_j[\beta]qB_\infty[\gamma],$$

$p \in \mathfrak{p}$, $q \in \mathfrak{q}$ and j sufficiently large.

Finally, we are going to prove the associativity of the operation $\circ$.

Proposition 2.8. *The product of double cosets is associative.*

Proof. Let $\alpha, \beta, \gamma, \delta \in \mathbb{N}$ and consider $\mathfrak{a} \in B_\infty[\alpha]\backslash B_\infty/B_\infty[\beta]$, $\mathfrak{b} \in B_\infty[\beta]\backslash B_\infty/B_\infty[\gamma]$ and $\mathfrak{c} \in B_\infty[\gamma]\backslash B_\infty/B_\infty[\delta]$. Choose representatives $a \in \mathfrak{a}$, $b \in \mathfrak{b}$ and $c \in \mathfrak{c}$ and consider $k = \max\{\alpha, \beta, \gamma, \delta, \operatorname{supp} a, \operatorname{supp} b, \operatorname{supp} c\} + 1$. Then

$$(\mathfrak{a}\mathfrak{b})\mathfrak{c} = B_\infty[\alpha]a\theta_k[\beta]b\theta_l[\gamma]cB_\infty[\delta] \qquad \text{and} \qquad \mathfrak{a}(\mathfrak{b}\mathfrak{c}) = B_\infty[\alpha]a\theta_{l'}[\beta]b\theta_k[\gamma]cB_\infty[\delta].$$

To prove our claim we are going to show that the double cosets above are the same, by exhibiting two representatives that are equal (Figs. 5 and 6 give an example of the process involved). Here we are assuming $\beta \leq \gamma$, the case $\gamma < \beta$ is analogous.

Throughout the rest of the proof we will use the symbol $a \equiv b$ to signify that a and b are representatives of the same double coset of $B_\infty[\alpha]\backslash B_\infty/B_\infty[\gamma]$, that is, we can find elements $h \in B_\infty[\alpha]$ and $k \in B_\infty[\gamma]$ such that $hak = b$.

Using the notation of Lemma 2.3 we can write

$$a\theta_k[\beta]b\theta_l[\gamma]c = a\begin{bmatrix} k+\beta & \to & \beta+1 \\ \downarrow & & \downarrow \\ 2k+\beta-1 & \to & k+\beta \end{bmatrix} b \begin{bmatrix} 2k+\beta+\gamma & \to & \gamma+1 \\ \downarrow & & \downarrow \\ 4k+2\beta+\gamma-1 & \to & 2k+\gamma+\beta \end{bmatrix} c,$$

$$a\theta_{l'}[\beta]b\theta_k[\gamma]c = a\begin{bmatrix} 2k+\gamma+\beta & \to & \beta+1 \\ \downarrow & & \downarrow \\ 4k+2\gamma+\beta-1 & \to & 2k+\gamma+\beta \end{bmatrix} b \begin{bmatrix} k+\gamma & \to & \gamma+1 \\ \downarrow & & \downarrow \\ 2k+\gamma-1 & \to & k+\gamma \end{bmatrix} c.$$

Using the same lemma, we can see that

$$\theta_k[\beta] = R_1P; \quad P = \begin{bmatrix} k+1 & \to & \beta+1 \\ \downarrow & & \downarrow \\ 2k & \to & k+\beta \end{bmatrix}, \qquad R_1 = \begin{bmatrix} k+\beta & \to & k+2 \\ \downarrow & & \downarrow \\ 2k+\beta-1 & \to & 2k+1 \end{bmatrix},$$

$$\theta_{l'}[\beta] = R_2P_2; \quad P_2 = \begin{bmatrix} k+1 & \to & \beta+1 \\ \downarrow & & \downarrow \\ 3k+\gamma & \to & 2k+\beta+\gamma \end{bmatrix}, \quad R_2 = \begin{bmatrix} 2k+\beta+\gamma & \to & k+2 \\ \downarrow & & \downarrow \\ 4k+2\gamma+\beta-1 & \to & 3k+\gamma+1 \end{bmatrix},$$

$$\theta_k[\gamma] = P_3R_3; \quad P_3 = \begin{bmatrix} k+\gamma & \to & \gamma+1 \\ \downarrow & & \downarrow \\ 2k & \to & k+1 \end{bmatrix}, \qquad R_3 = \begin{bmatrix} 2k+1 & \to & k+2 \\ \downarrow & & \downarrow \\ 2k+\gamma-1 & \to & k+\gamma \end{bmatrix},$$

$$\theta_l[\gamma] = P_4R_4; \quad P_4 = \begin{bmatrix} 2k+\beta+\gamma & \to & \gamma+1 \\ \downarrow & & \downarrow \\ 3k+\beta & \to & k+1 \end{bmatrix}, \quad R_4 = \begin{bmatrix} 3k+\beta & \to & k+2 \\ \downarrow & & \downarrow \\ 4k+2\beta+\gamma-1 & \to & 2k+\beta+\gamma \end{bmatrix}.$$

Since $R_i \in B_\infty[k+1]$, $1 \le i \le 4$, we have

$$aR_1PbP_4R_4c = R_1aPbP_4cR_4 \equiv aPbP_4c, \qquad aR_2P_2bP_3R_3c = R_2aP_2bP_3cR_3 \equiv aP_2bP_3c.$$

Notice also that $P_4 = R_5W$, where

$$R_5 = \begin{bmatrix} 2k+\beta+\gamma & \to & 2k+2 \\ \downarrow & & \downarrow \\ 3k+\beta & \to & 3k-\gamma+2 \end{bmatrix} \quad \text{and} \quad W = \begin{bmatrix} 2k+1 & \to & \gamma+1 \\ \downarrow & & \downarrow \\ 3k-\gamma+1 & \to & k+1 \end{bmatrix}.$$

Since $\operatorname{supp} P = 2k$ and $R_5 \in B_\infty[2k+1]$, $R_5P = PR_5$ and we have $aPbR_5Wc = aPR_5bWc = R_5aPbWc \equiv aPbWc$.

Our next objective is to find elements $E, A \in B_\infty$ such that $aP_2bP_3c \equiv aPbEAWc$.

Step 1. $aP_2bP_3c = aPbELP_3c$. Consider the element

$$F = \begin{bmatrix} 2k+1 & \to & k+\beta+1 \\ \downarrow & & \downarrow \\ 3k+\gamma & \to & 2k+\beta+\gamma \end{bmatrix},$$

and notice that $P_2 = PF$. Since $F \in B_\infty[k]$, we see that $bF = Fb$.

Moreover, $F = EL$ where

$$E = \begin{bmatrix} 2k+1 & \to & k+\beta+1 \\ \downarrow & & \downarrow \\ 2k-\beta+\gamma & \to & k+\gamma \end{bmatrix} \quad \text{and} \quad L = \begin{bmatrix} 2k-\beta+\gamma+1 & \to & k+\gamma+1 \\ \downarrow & & \downarrow \\ 3k+\gamma & \to & 2k+\beta+\gamma \end{bmatrix}.$$

Step 2. $LP_3c \equiv CP_3c$ for some C. In fact, consider

$$C = \begin{bmatrix} 2k+\gamma-\beta+1 & \to & k+\gamma+1 \\ \downarrow & & \downarrow \\ 3k-\beta+1 & \to & 2k+1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 3k-\beta+2 & \to & 2k+2 \\ \downarrow & & \downarrow \\ 3k+\gamma & \to & 2k+\beta+\gamma \end{bmatrix}.$$

Then $L = CD$ and, since $D \in B_\infty[2k+1]$ and $\operatorname{supp} P_3 = 2k$, we have $DP_3c = P_3cD \equiv P_3c$. Hence $LP_3c \equiv CP_3c$.

Step 3. $CP_3 = AW$ for some A. In fact, consider $A = \begin{bmatrix} 2k+\gamma-\beta+1 & \to & 2k+2 \\ \downarrow & & \downarrow \\ 3k-\beta+1 & \to & 3k-\gamma+2 \end{bmatrix}$.

Then,

$$
\begin{aligned}
CP_3 &= \begin{bmatrix} 2k+\gamma-\beta+1 & \to & k+\gamma+1 \\ \downarrow & & \downarrow \\ 3k-\beta+1 & \to & 2k+1 \end{bmatrix} \begin{bmatrix} k+\gamma & \to & \gamma+1 \\ \downarrow & & \downarrow \\ 2k & \to & k+1 \end{bmatrix} \\
&= \begin{bmatrix} 2k+\gamma-\beta+1 & \to & 2k+2 \\ \downarrow & & \downarrow \\ 3k-\beta+1 & \to & 3k-\gamma+2 \end{bmatrix} \begin{bmatrix} 2k+1 & \to & k+\gamma+1 \\ \downarrow & & \downarrow \\ 3k-\gamma+1 & \to & 2k+1 \end{bmatrix} \begin{bmatrix} k+\gamma & \to & \gamma+1 \\ \downarrow & & \downarrow \\ 2k & \to & k+1 \end{bmatrix} \\
&= \begin{bmatrix} 2k+\gamma-\beta+1 & \to & 2k+2 \\ \downarrow & & \downarrow \\ 3k-\beta+1 & \to & 3k-\gamma+2 \end{bmatrix} \begin{bmatrix} 2k+1 & \to & \gamma+1 \\ \downarrow & & \downarrow \\ 3k-\gamma+1 & \to & k+1 \end{bmatrix} = AW.
\end{aligned}
$$

Therefore, $aP_2bP_3c = aPbELP_3c \equiv aPbECP_3c = aPbEAWc$. At last, consider

$$
\tilde{W} = \begin{bmatrix} 3k-\beta+2 & \to & k+2 \\ \downarrow & & \downarrow \\ 4k-2\beta+\gamma+1 & \to & 2k-\beta+\gamma+1 \end{bmatrix}.
$$

Then $AW\tilde{W} = \theta_r[\gamma]$ with $r = 2k-\beta+1$. Hence $aPbEAWc \equiv aPbE\theta_r[\gamma]c$ and, by Lemma 2.5, $E\theta_r[\gamma] = \theta_r[\gamma]C_r(E)$. Therefore,

$$
aPbE\theta_r[\gamma]c = aPb\theta_r[\gamma]C_r(E)c = aPb\theta_r[\gamma]cC_r(E) \equiv aPb\theta_r[\gamma]c \equiv aPbAWc.
$$

Furthermore, since $A \in B_\infty[2k+1]$ and $\operatorname{supp} P = 2k$,

$$
aPbAWc = aPAbWc = AaPbWc \equiv aPbWc. \qquad \blacksquare
$$

Example 2.9. In this example we illustrate the method described in the proof of the theorem above. Here we used $a = \sigma_2^{-1}\sigma_1^{-1}$, $b = \sigma_1^2$, $c = \sigma_1^2\sigma_2^2$, $\alpha = \delta = 3$, $\beta = 1$ and $\gamma = 2$. In each of the figures below, the diagrams are different representatives of the same double coset, obtained following the steps of the proof of Proposition 1.5. The dashed horizontal lines highlight the different braids mentioned in the captions.

2.2 Proof of Proposition 1.7

We show that the conjugacy problem for B_∞ can be reduced to a conjugacy problem in B_n, for some n. Given two braids $x, y \in B_\infty$, since these braids are finitely supported, there exists $n \in \mathbb{N}$ such that we can consider these braids as elements of B_n. Since the conjugacy problem has a solution in B_n, to prove the proposition it suffices to show that x is conjugate to y in B_n if and only if they are conjugate in B_∞. But this follows from the properties of the direct limit and the fact that the inclusions $i_n\colon B_n \to B_{n+1}$ do not merge conjugacy classes (see [7]).

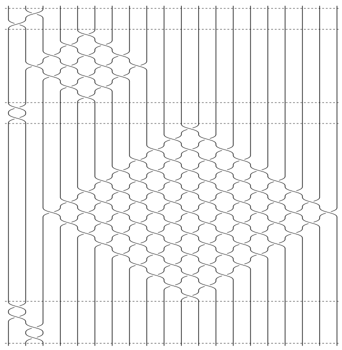
2.3 Proof of Theorem 1.9

Let p and q be representatives of the double cosets $\mathfrak{p} \in G[n]\backslash \mathrm{GL}(\infty)/G[k]$ and $\mathfrak{q} \in G[k]\backslash \mathrm{GL}(\infty)/G[m]$, respectively. Define the sequence of double cosets

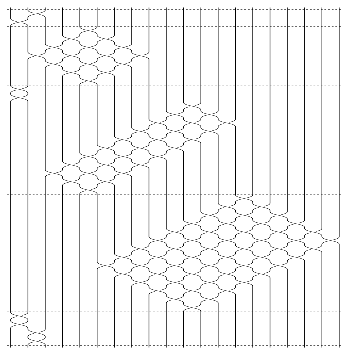
$$
\mathfrak{r}_j = G[n]p\Theta_j[k]qG[m],
$$

in $G[n]\backslash \mathrm{GL}(\infty)/G[m]$.

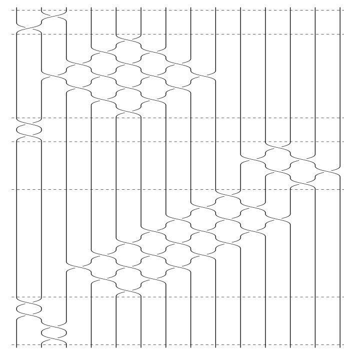
We remark the following identity:



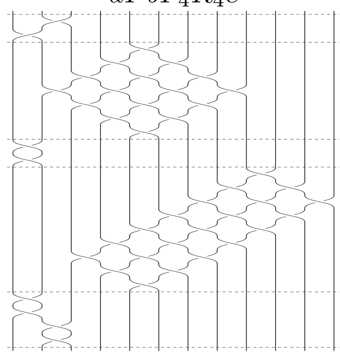
$a\theta_k[\beta]b\theta_l[\gamma]c$



$aPbP_4R_4c$



$aPbR_5Wc$



$aPbWc$

Figure 5. The equality $a\theta_k[\beta]b\theta_l[\gamma]c = aPbWc$.

Lemma 2.10. *If $\eta \colon B_\infty \to \mathrm{GL}(\infty)$ is the Burau representation, as defined in Section 1.3, the following identity holds*

$$\Theta_j[k] = \eta(\theta_j[k]), \qquad \textit{for all} \quad j, k \in \mathbb{N}.$$

Proposition 2.11. *The sequence $\mathfrak{r}_j$ above is eventually constant and its limit does not depend on the choice of representatives.*

Proof. Let $N \in \mathbb{N}$ be such that $N > \max\{m, k, n\}$ and p and q can be written as square $(k + N + \infty)$-matrices with the following block configuration:

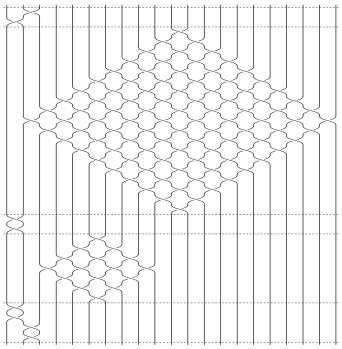
$$p = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1_\infty \end{pmatrix}, \qquad q = \begin{pmatrix} x & y & 0 \\ z & w & 0 \\ 0 & 0 & 1_\infty \end{pmatrix}.$$

Suppose that for some $i \geq N$ we have $\mathfrak{r}_i = \mathfrak{r}_N$. We show that $\mathfrak{r}_i = \mathfrak{r}_{i+1}$. As we saw in Proposition 2.2, there are elements $u, l \in B_\infty$ such that $\theta_i[k] = u\theta_{i+1}[k]l$. Hence, if $U = \eta(u)$ and $L = \eta(l)$, we have

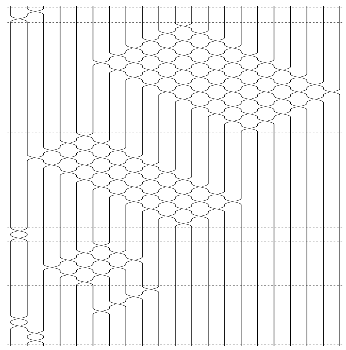
$$\Theta_i[k] = U\Theta_{i+1}[k]L.$$

Furthermore, U and L have the following block configuration

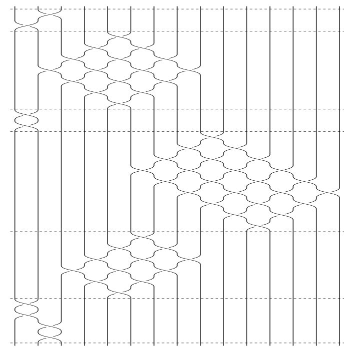
$$U = \begin{pmatrix} 1_k & 0 & 0 & 0 \\ 0 & 1_i & 0 & 0 \\ 0 & 0 & \nu & 0 \\ 0 & 0 & 0 & 1_\infty \end{pmatrix}, \qquad L = \begin{pmatrix} 1_k & 0 & 0 & 0 \\ 0 & 1_i & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 1_\infty \end{pmatrix}.$$



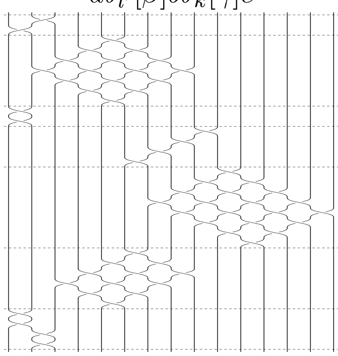
$a\theta_{l'}[\beta]b\theta_k[\gamma]c$



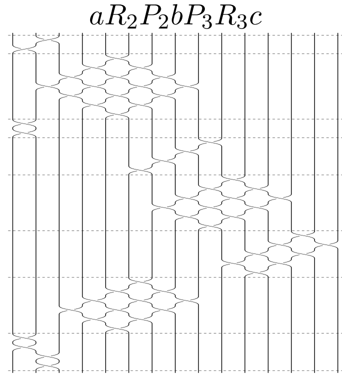
$aR_2P_2bP_3R_3c$



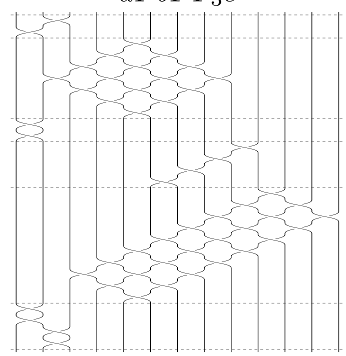
$aPbFP_3c$



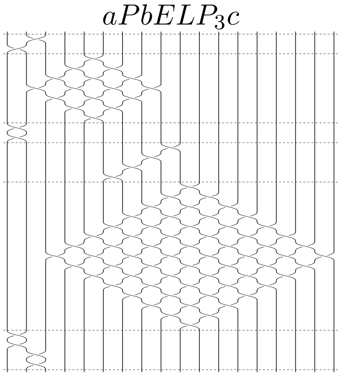
$aPbELP_3c$



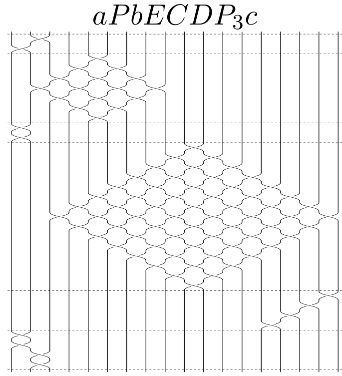
$aPbECDP_3c$



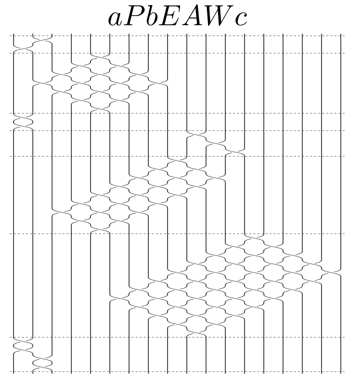
$aPbEAWc$



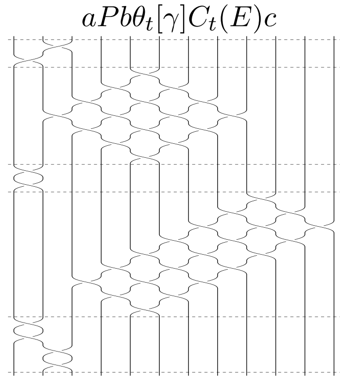
$aPbE\theta_t[\gamma]c$



$aPb\theta_t[\gamma]C_t(E)c$



$aPbAW\tilde{W}c$



$aPbWc$

Figure 6. The equality $a\theta_{l'}[\beta]b\theta_k[\gamma]c = aPbWc$.

Thus,

$$Up = pU \qquad \text{and} \qquad Lq = qL.$$

Consequently,

$$p\Theta_i[k]q = pU\Theta_{i+1}[k]Lq = Up\Theta_{i+1}[k]qL.$$

Since U and L are elements of the image of the Burau representation η, we have that $U, L \in G[k]$ and therefore

$$\mathfrak{r}_{i+1} = G[n]p\Theta_{i+1}[k]qG[m] = G[n]Up\Theta_{i+1}[k]qLG[m] = G[n]p\Theta_i[k]qG[m] = \mathfrak{r}_i.$$

To show that the limit of the sequence $\mathfrak{r}_i$ does not depend on the choice of representatives it suffices to show that, for any H and J in $G[k]$, we have

$$\lim G[n]p\Theta_i[k]qG[m] = \lim G[n]pJ\Theta_i[k]HqG[m].$$

Let $N > 0$ be as before. Consider $M > N$ such that H and J are square $(k + M + \infty)$-matrices with the block configuration:

$$H = \begin{pmatrix} 1_k & 0 & 0 \\ 0 & h & 0 \\ 0 & 0 & 1_\infty \end{pmatrix}, \qquad J = \begin{pmatrix} 1_k & 0 & 0 \\ 0 & j & 0 \\ 0 & 0 & 1_\infty \end{pmatrix}.$$

Since H preserves the vector v, we have that $V_M h = V_M$. Similarly, $jV_M = V_M$. Therefore,

$$\begin{aligned} J\Theta_M[k]H &= \begin{pmatrix} 1_k & 0 & 0 & 0 \\ 0 & j & 0 & 0 \\ 0 & 0 & 1_M & 0 \\ 0 & 0 & 0 & 1_\infty \end{pmatrix} \begin{pmatrix} 1_k & 0 & 0 & 0 \\ 0 & V_M & t^M 1_M & 0 \\ 0 & 1_M & 0 & 0 \\ 0 & 0 & 0 & 1_\infty \end{pmatrix} \begin{pmatrix} 1_k & 0 & 0 & 0 \\ 0 & h & 0 & 0 \\ 0 & 0 & 1_M & 0 \\ 0 & 0 & 0 & 1_\infty \end{pmatrix} \\ &= \begin{pmatrix} 1_k & 0 & 0 & 0 \\ 0 & jV_M h & t^M h & 0 \\ 0 & j & 0 & 0 \\ 0 & 0 & 0 & 1_\infty \end{pmatrix} = \begin{pmatrix} 1_k & 0 & 0 & 0 \\ 0 & V_M & t^M h & 0 \\ 0 & j & 0 & 0 \\ 0 & 0 & 0 & 1_\infty \end{pmatrix} \\ &= \begin{pmatrix} 1_k & 0 & 0 & 0 \\ 0 & 1_M & 0 & 0 \\ 0 & 0 & h & 0 \\ 0 & 0 & 0 & 1_\infty \end{pmatrix} \begin{pmatrix} 1_k & 0 & 0 & 0 \\ 0 & V_M & t^M 1_M & 0 \\ 0 & 1_M & 0 & 0 \\ 0 & 0 & 0 & 1_\infty \end{pmatrix} \begin{pmatrix} 1_k & 0 & 0 & 0 \\ 0 & 1_M & 0 & 0 \\ 0 & 0 & j & 0 \\ 0 & 0 & 0 & 1_\infty \end{pmatrix}. \end{aligned}$$

Call J' the new matrix containing the block j and H' the new matrix containing the block h. Then we have

$$pJ\Theta_M[k]Hq = pH'\Theta_M[k]J'q = H'p\Theta_M[k]qJ'.$$

Therefore, $p\Theta_M[k]q$ and $pJ\Theta_M[k]Hq$ belong to the same double coset for M sufficiently large. ■

Therefore, we have a well-defined product of the double cosets $\mathfrak{p}$ and $\mathfrak{q}$ given by

$$\mathfrak{p} \star_t \mathfrak{q} = \lim G[n]p\Theta_j[k]qG[m].$$

Proposition 2.12. *The operation defined above is associative. Furthermore, the Burau representation is a functor between the categories of double cosets of* $\mathrm{GL}(\infty)$ *and of* B_∞.

Proof. The proof of the associative property is analogous to the proof of Theorem 2.8, using Lemma 2.10. The functoriality follows from Lemma 2.10. ■

3 Connections with other direct limits of groups

We can extend the above constructions to the product $G^{[n]} = B_\infty \times \cdots \times B_\infty$ of n copies of the infinite braid group. Let K be the diagonal subgroup of $G^{[n]}$. Clearly, K is naturally isomorphic to B_∞. Let $K[\alpha]$ be the image of $B_\infty[\alpha]$ under this isomorphism. We define the product of double cosets componentwise.

Corollary 3.1. *Consider two double cosets*

$$\mathfrak{p} \in K[\alpha]\backslash G^{[n]}/K[\beta], \qquad \mathfrak{q} \in K[\beta]\backslash G^{[n]}/K[\gamma],$$

and let p and q be their respective representatives. Then the operation given by

$$\mathfrak{p} \circ \mathfrak{q} = K[\alpha]p\theta_j[\beta]qK[\beta],$$

for j sufficiently large, is well-defined and associative.

Proof. It follows from Propositions 2.2, 2.7 and 2.8. ■

When there is a surjective homomorphism B_∞ onto a group A, we have an induced operation on the double cosets of A. More precisely, let $\psi\colon B_\infty \to A$ be a surjective homomorphism and consider, for each $\alpha \in \mathbb{N}$, the image $A[\alpha]$ of the subgroup $B_\infty[\alpha]$ by ψ. Then the induced product on the double cosets of A with relation to the subgroups $A[\alpha]$ is well-defined. Indeed, this follows from the fact that the sequence used to define the product of double cosets in B_∞ not only converges, it becomes constant.

Let S_∞ denote the infinite symmetric group, that is, the set of permutations $s\colon \mathbb{N}^* \to \mathbb{N}^*$ such that $s(i) = i$ for all but finitely many $i \in \mathbb{N}^*$ equipped with the composition of functions. Consider the subgroups $S_\infty[\alpha] \subset S_\infty$, $\alpha \in \mathbb{N}$ consisting of the elements fixing the set $\{1, 2, \ldots, \alpha\}$ pointwise. In [12], Neretin defined an operation on the double cosets of S_∞ with relation to the subgroups $S_\infty[\alpha]$ as follows: For integers $\beta \geq 0$ and $n > 0$, denote by $\theta_n^s[\beta]$ the permutation given by

$$\theta_n^s[\beta](i) = \begin{cases} i+n, & \beta < i \leq \beta + n, \\ i-n, & \beta + n < i \leq \beta + 2n, \\ i, & \text{otherwise.} \end{cases}$$

Given double cosets

$$\mathfrak{p} \in S_\infty[\alpha]\backslash S_\infty/S_\infty[\beta] \qquad \text{and} \qquad S_\infty[\beta]\backslash S_\infty/S_\infty[\gamma],$$

and their representatives $p \in \mathfrak{p}$ and $q \in \mathfrak{q}$, define their product as

$$\mathfrak{p} \circ^s \mathfrak{q} = S_\infty[\alpha]p\theta_n^s[\beta]qS_\infty[\gamma],$$

for sufficiently large n.

On the other hand, consider the canonical homomorphism $j\colon B_\infty \to S_\infty$ that associates to each braid the corresponding permutation of its endpoints. It is clear that this is a surjective homomorphism (it is, up to conjugacy, the only surjective homomorphism from B_∞ to S_∞, see [1]) and hence induces an operation on the double cosets of S_∞ with relation to the subgroups $j(B_\infty[\alpha])$, $\alpha \in \mathbb{N}$. Furthermore, it is easy to check that $j(B_\infty[\alpha]) = S_\infty[\alpha]$ for each $\alpha \in \mathbb{N}$.

Proposition 3.2. *The operation on double cosets of S_∞, with relation to the subgroups $S_\infty[\alpha]$, $\alpha \in \mathbb{N}$, coincides with the operation induced by the group B_∞.*

Proof. In fact, it suffices to check that the elements $\theta_n^s[\beta]$ and $j(\theta_n[\beta])$ coincide for all integers $n > 0$ and $\beta \geq 0$. But this identity follows directly from the definition of $\theta_n[\beta]$ and j. ■

As a last remark, we point out some similarities between the multiplicative structure defined in B_∞ and that of $\mathrm{Aut}(F_\infty)$. The group $\mathrm{Aut}(F_\infty)$ is defined as follows: Let F_n be the free group with n generators $x_1, \dots, x_n$ and denote by $Aut(F_n)$ the group of automorphisms of F_n. Then

$$\mathrm{Aut}(F_\infty) = \lim \mathrm{Aut}(F_n).$$

The limit is taken with relation to the obvious inclusion $\mathrm{Aut}(F_n) \to \mathrm{Aut}(F_{n+1})$.

For each $\alpha \in \mathbb{N}$ consider the subgroup $H(\alpha)$ of $\mathrm{Aut}(F_\infty)$ of automorphisms h such that $h(x_i) = x_i$ for $i \leq \alpha$. In [13], it is defined a product on the double cosets of $\mathrm{Aut}(F_\infty)$ in the following way: Consider the automorphism $\vartheta_j[\beta] \in \mathrm{Aut}(F_\infty)$ given by

$$\vartheta_j[\beta](x_i) = \begin{cases} x_i, & i \leq \beta, i > 2j + \beta, \\ x_{i+j}, & \beta < i \leq \beta + j, \\ x_{i-j}, & \beta + j < i \leq \beta + 2j. \end{cases}$$

Then, for p and q in $\mathrm{Aut}(F_\infty)$, the product of the double cosets $H(\alpha)pH(\beta)$ and $H(\beta)qH(\gamma)$ is the double coset limit of the sequence $p\vartheta_j[m]q$ in $H(\alpha)\backslash \mathrm{Aut}(F_\infty)/H(\gamma)$.

For each $n \in \mathbb{N}$, let $i_n \colon B_n \to \mathrm{Aut}(F_n)$ be the *Artin representation* of B_n on the free group F_n, given by

$$i_n(\sigma_j)(x_k) = \begin{cases} x_j, & k = j + 1, \\ x_j x_{j+1} x_j^{-1}, & k = j, \\ x_k, & \text{otherwise}. \end{cases}$$

This representation is faithful and therefore we can identify B_n with the image of i_n in $\mathrm{Aut}(F_n)$. Consider the limit homomorphism $i_\infty \colon B_\infty \to \mathrm{Aut}(F_\infty)$. The element $\vartheta_j[m]$ is related to the image of the element $\theta_j[m]$ as we see in the following proposition

Proposition 3.3. *Let β be a fixed positive integer. For each $k \in \mathbb{N}$, consider the element $y_k = x_{\beta+k} x_{\beta+k-1} \cdots x_{\beta+1} \in F_\infty$. Then*

$$i_\infty(\theta_k[\beta])(x_i) = \begin{cases} x_i, & i \leq \beta, i > 2k + \beta, \\ y_k^{-1} x_{i+k} y_k, & \beta + 1 \leq i \leq k + \beta, \\ x_{i-k}, & k + \beta < i \leq 2k + \beta. \end{cases}$$

In other words,

$$i_\infty(\theta_k[\beta])(x_i) = \begin{cases} y_k^{-1} \vartheta_k[\beta](x_i) y_k, & \beta + 1 \leq i \leq k + \beta, \\ \vartheta_k[\beta](x_i), & \textit{otherwise}. \end{cases}$$

Proof. For $k = 1$ we have that $\theta_1[\beta] = \sigma_{\beta+1}$ and therefore

$$i_\infty(\theta_1[\beta])(x_i) = i_\infty(\sigma_{\beta+1})(x_i) = \begin{cases} x_i, & i \leq \beta, i > \beta + 2, \\ x_i^{-1} x_{i+1} x_i, & i = 1 + \beta, \\ x_{i-1}, & i = 2 + \beta. \end{cases}$$

We are going to show the truth of the identity by induction on k. Suppose the identity holds for k. We can write $\theta_{k+1}[\beta]$ as

$$\theta_{k+1}[\beta] = \sigma_{k+\beta+1} \cdots \sigma_{2k+\beta+1} \theta_k[\beta] \sigma_{2k+\beta} \cdots \sigma_{k+\beta+1}.$$

If we put $w = \sigma_{k+\beta+1} \cdots \sigma_{2k+\beta+1}$ and $s = \sigma_{2k+\beta} \cdots \sigma_{k+\beta+1}$ we can re-write the equation above as

$$\theta_{k+1}[\beta] = w\theta_k[\beta]s.$$

We have five cases to analyze:

Case 1. When $\beta + 1 \leq i \leq k + \beta$, notice that $i_\infty(s)(x_i) = x_i$ and $i_\infty(\theta_k[\beta])(x_i) = y_k^{-1} x_{i+k} y_k$, therefore $i_\infty(\theta_{k+1}[\beta])(x_i) = i_\infty(w)\big(y_k^{-1} x_{i+k} y_k\big)$. Now,

$$\begin{aligned} i_\infty(w)(x_{i+k}) &= i_\infty(\sigma_{k+\beta+1} \cdots \sigma_{i+k-1}) i_\infty(\sigma_{i+k})(x_{i+k}) \\ &= i_\infty(\sigma_{k+\beta+1} \cdots \sigma_{i+k-2}) i_\infty(\sigma_{i+k-1})\big(x_{i+k}^{-1} x_{i+k+1} x_{i+k}\big) \\ &= i_\infty(\sigma_{k+\beta+1} \cdots \sigma_{i+k-3}) i_\infty(\sigma_{i+k-2})\big(x_{i+k-1}^{-1} x_{i+k+1} x_{i+k-1}\big) = \cdots \\ &= i_\infty(\sigma_{k+\beta+1})\big(x_{k+\beta+2}^{-1} x_{k+i+1} x_{k+\beta+2}\big) = x_{k+\beta+1}^{-1} x_{k+i+1} x_{k+\beta+1}. \end{aligned}$$

Hence,

$$i_\infty(w)\big(y_k^{-1} x_{i+k} y_k\big) = y_k^{-1} i_\infty(w)(x_{i+k}) y_k = y_k^{-1} x_{k+\beta+1}^{-1} x_{k+i+1} x_{k+\beta+1} y_k = y_{k+1}^{-1} x_{k+i+1} y_{k+1}.$$

Case 2. When $i = k + \beta + 1$, we have that

$$i_\infty(s)(x_{k+i+\beta}) = x_{k+\beta+1}^{-1} \cdots x_{2k+\beta}^{-1} x_{2k+\beta+1} x_{2k+\beta} \cdots x_{k+\beta+1}.$$

Hence,

$$\begin{aligned} i_\infty(\theta_k[\beta]s)(x_{k+\beta+1}) &= i_\infty(\theta_k[\beta])\big(x_{k+\beta+1}^{-1} \cdots x_{2k+\beta}^{-1} x_{2k+\beta+1} x_{2k+\beta} \cdots x_{k+\beta+1}\big) \\ &= x_{\beta+1}^{-1} \cdots x_{k+\beta}^{-1} x_{2k+\beta+1} x_{k+\beta} \cdots x_{\beta+1} = y_k^{-1} x_{2k+\beta+1} y_k. \end{aligned}$$

Furthermore, $i_\infty(w)(x_{2k+\beta+1}) = x_{k+\beta+1}^{-1} x_{2k+\beta+2} x_{k+\beta+1}$ and hence

$$\begin{aligned} i_\infty(\theta_{k+1}[\beta])(x_{k+\beta+1}) &= i_\infty(w)\big(y_k^{-1} x_{2k+\beta+1} y_k\big) \\ &= y_k^{-1} x_{k+\beta+1}^{-1} x_{2k+\beta+2} x_{k+\beta+1} y_k = y_{k+1}^{-1} x_{2k+\beta+2} y_{k+1}. \end{aligned}$$

Case 3. When $k + \beta + 1 < i \leq 2k + \beta + 1$, it is sufficient to notice that $i_\infty(s)(x_i) = x_{i-1}$, $i_\infty(\theta_k[\beta])(x_{i-1}) = x_{i-k-1}$ and $i_\infty(w)(x_{i-k-1}) = x_{i-k-1}$.

Case 4. For the case $i = 2k + \beta + 2$ we have $i_\infty(\theta_k[\beta]s)(x_{2k+\beta+2}) = x_{2k+\beta+2}$. Furthermore, $i_\infty(w)(x_{2k+\beta+2}) = x_{k+\beta+1}$.

Case 5. For $i \leq \beta$ or $i > 2k + \beta + 2$, we have that $i_\infty(w)(x_i) = i_\infty(\theta_k[\beta](x_i) = i_\infty(s)(x_i) = x_i$ and the result follows. ■

Thus the elements $\vartheta_j[\beta]$ and $i_\infty(\theta_j[\beta])$ are always conjugate in $\mathrm{Aut}(F_\infty)$ (in particular, by an element of $H(\beta)$). Nevertheless i_∞ does not induce a homomorphism between the monoids of double cosets. In fact, consider the braid $\omega = \sigma_2^{-1} \sigma_3 \sigma_1 \sigma_3 \sigma_2$ in B_∞ and its projection $[\omega]$ in $B[2] \backslash B_\infty / B[2]$. Then $i_\infty(\omega \theta_N[2] \omega)$ and $i_\infty(\omega) \vartheta_N[2] i_\infty(\omega)$ do not belong to the same double coset of $H(2) \backslash \mathrm{Aut}(F_\infty)/H(2)$.

Acknowledgements

This research was supported by FAPESP process 2015/03341-9. We are indebted to L. Funar and A.K.M. Libardi for useful discussions and continuous support.

References

[1] Artin E., Braids and permutations, *Ann. of Math.* **48** (1947), 643–649.

[2] Artin E., Theory of braids, *Ann. of Math.* **48** (1947), 101–126.

[3] Birman J.S., Braids, links, and mapping class groups, *Annals of Mathematics Studies*, Vol. 82, Princeton University Press, Princeton, N.J., University of Tokyo Press, Tokyo, 1974.

[4] Birman J.S., Brendle T.E., Braids: a survey, in Handbook of Knot Theory, Elsevier B.V., Amsterdam, 2005, 19–103, math.GT/0409205.

[5] Dehornoy P., A fast method for comparing braids, *Adv. Math.* **125** (1997), 200–235.

[6] Geck M., Pfeiffer G., Characters of finite Coxeter groups and Iwahori–Hecke algebras, *London Mathematical Society Monographs New Series*, Vol. 21, The Clarendon Press, Oxford University Press, New York, 2000.

[7] González-Meneses J., Geometric embeddings of braid groups do not merge conjugacy classes, *Bol. Soc. Mat. Mex.* **20** (2014), 297–305.

[8] Neretin Yu.A., Categories of symmetries and infinite-dimensional groups, *London Mathematical Society Monographs, New Series*, Vol. 16, The Clarendon Press, Oxford University Press, New York, 1996.

[9] Neretin Yu.A., On multiplication of double cosets for $\mathrm{GL}(\infty)$ over a finite field, arXiv:1310.1596.

[10] Neretin Yu.A., Sphericity and multiplication of double cosets for infinite-dimensional classical groups, *Funct. Anal. Appl.* **45** (2011), 225–239, arXiv:1101.4759.

[11] Neretin Yu.A., Infinite tri-symmetric group, multiplication of double cosets, and checker topological field theories, *Int. Math. Res. Not.* **2012** (2012), 501–523, arXiv:0909.4739.

[12] Neretin Yu.A., Infinite symmetric groups and combinatorial constructions of topological field theory type, *Russian Math. Surveys* **70** (2015), 715–773, arXiv:1502.03472.

[13] Neretin Yu.A., Several remarks on groups of automorphisms of free groups, *J. Math. Sci.* **215** (2016), 748–754, arXiv:1306.6035.

[14] Neretin Yu.A., Combinatorial encodings of infinite symmetric groups and descriptions of semigroups of double cosets, *J. Math. Sci.* **232** (2018), 138–156, arXiv:1106.1161.

[15] Okounkov A.Yu., Thoma's theorem and representations of the infinite bisymmetric group, *Funct. Anal. Appl.* **28** (1994), 100–107.

[16] Ol'shankii G.I., Infinite-dimensional classical groups of finite R-rank: description of representations and asymptotic theory, *Funct. Anal. Appl.* **18** (1984), 22–34.

[17] Ol'shankii G.I., Unitary representations of the group $\mathrm{SO}_{\mathrm{o}}(\infty,\infty)$ as limits of unitary representations of the groups $\mathrm{SO}_{\mathrm{o}}(n,\infty)$ as $n\to\infty$, *Funct. Anal. Appl.* **20** (1986), 292–301.

[18] Ol'shankii G.I., Method of holomorphic extensions in the theory of unitary representations of infinite-dimensional classical groups, *Funct. Anal. Appl.* **22** (1988), 273–285.

[19] Ol'shankii G.I., Unitary representations of (G,K)-pairs connected with the infinite symmetric group $S(\infty)$, *Leningrad Math. J.* **1** (1990), 983–1014.

[20] Ol'shankii G.I., Unitary representations of infinite-dimensional pairs (G,K) and the formalism of R. Howe, in Representation of Lie Groups and Related Topics, *Adv. Stud. Contemp. Math.*, Vol. 7, Gordon and Breach, New York, 1990, 269–463.

[21] Ol'shankii G.I., On semigroups related to infinite-dimensional groups, in Topics in representation theory, *Adv. Soviet Math.*, Vol. 2, Amer. Math. Soc., Providence, RI, 1991, 67–101.