ETH Zürich FS 2013
D-MATH
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Applied Stochastic Processes

Exercise Sheet 2

Please hand in by 13:00 on Friday 22-Mar-2013 in the assistant's box in HG E 65.

Exercise 2-1

Let $X_1, \dots, X_n$ be real-valued i.i.d. random variables with a density f. Denote by $X_{(1)}, \dots, X_{(n)}$ the *order statistics* of $X_1, \dots, X_n$, i.e.

$$X_{(1)} := \min\{X_1, \dots, X_n\}, X_{(2)} := \min(\{X_1, \dots, X_n\} \backslash \{X_{(1)}\}),$$
$$X_{(3)} := \min(\{X_1, \dots, X_n\} \backslash \{X_{(1)}, X_{(2)}\}), \dots$$

Show that the joint density g of $X_{(1)}, \dots, X_{(n)}$ is given by

$$g(x_1, \dots, x_n) = n! \prod_{i=1}^{n} f(x_i) \mathbb{1}_{\{x_1 < x_2 < \dots < x_n\}}.$$

Exercise 2-2

Suppose you live on a busy street, and there are taxis passing by quite frequently. Instead of booking one, you walk out on the street and wait for the next taxi. We shall model the number of taxi arrivals as a Poisson process $(N_t)_{t \geq 0}$ with constant rate $\lambda > 0$ and successive jump times $(S_k)_{k \in \mathbb{N}_0}$ (with $S_0 = 0$). We define the following random variables:

$$\gamma_t = S_{N_t+1} - t \quad \text{(waiting time for the next taxi)},$$
$$\delta_t = t - S_{N_t} \quad \text{(time since the previous taxi passed by, or since time 0)},$$
$$\beta_t = \gamma_t + \delta_t \,.$$

a) Determine the distribution of the random variables γ_t and δ_t.

b) Determine the joint distribution of γ_t and δ_t and explain the result.

c) Compute the distribution of the random variable β_t.

d) Compute $E[\beta_t]$ and $\lim_{t \to \infty} E[\beta_t]$. Compare the expectation of β_t with that of the inter-arrival times T_i. Give an interpretation of your results.

Exercise 2-3

A radioactive source Q emits on average λ particles per second and a fraction $p \in (0,1)$ of all emitted particles hits a detector. Let N_t denote the number of particles which the source emits in the time period $[0,t]$. We assume that $(N_t)_{t\geq 0}$ is a Poisson process with a constant rate $\lambda > 0$. Fix $t > 0$ and let $N_Q := N_t$ denote the number of particles emitted by the source Q in the time period $[0,t]$ and N_D the number of particles which hit the detector. We define $N_X := N_Q - N_D$.

a) Compute the conditional distribution of N_D given N_Q and the conditional distribution of N_X given N_Q.

b) Determine the joint distribution of N_D and N_X. Are N_D and N_X independent?

Exercise 2-4

Suppose you have just finished writing your bachelor's thesis with 60 pages. Applying your knowledge from ASP, you assume that as a function of the number of pages, the number of typos in your thesis follows a Poisson process with some unknown rate $\lambda > 0$ per page. You give your thesis to two of your friends, who independently proofread it. The first one finds 36 typos, the second one 30 typos, and their lists contain 18 typos in common.

a) Reckoning that friends 1 and 2 find each typo independently with unknown probabilities $p_1 \in (0,1)$ and $p_2 \in (0,1)$ respectively, calculate estimates of p_1, p_2, λ and the expected number of undiscovered typos.

b) Dissatisfied with the estimated expected number of undiscovered typos, you decide to give the first ℓ pages of your thesis, $1 \leq \ell \leq 60$, to a professional proofreader, who finds typos with a probability of 99% and charges 0.5 CHF per page. How much money do you have to spend in order to have an estimated expected number of at most 3 undiscovered typos in your thesis?

Exercise 2-5

Let $N = (N_t)_{t\geq 0}$ be an inhomogeneous Poisson process with rate $\rho(t) = \alpha t$, where α is a positive constant. Denote by $(S_k)_{k\in\mathbb{N}}$ the sequence of successive jump times of N.

a) Show that $S_k < \infty$ a.s. for all $k \in \mathbb{N}$ and determine the distribution of S_k.

b) For $k \in \mathbb{N}$ set $T_k := S_k - S_{k-1}$, where $S_0 = 0$. Show that the T_k are not independent by computing the joint distribution of T_1 and T_2.