ETH Zürich FS 2013
D-MATH
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Applied Stochastic Processes

Exercise Sheet 3

Please hand in by 13:00 on Friday 12-Apr-2013 in the assistant's box in HG E 65.

Exercise 3-1

Let $(N_t)_{t\geq 0}$ be a renewal process with interarrival times having the density

$$f(x) = \lambda^2 x \mathrm{e}^{-\lambda x} \mathbb{1}_{\{x\geq 0\}}, \quad \lambda > 0.$$

Compute the renewal function $M(t) := E[N_t]$ and its asymptotic growth rate $\lim\limits_{t\to\infty} \dfrac{M(t)}{t}$.

Exercise 3-2

Let $(U_i, V_i)_{i\in\mathbb{N}}$ be a sequence of i.i.d. random vectors with $U_i \geq 0$, $V_i \geq 0$. Assume that $T_i = U_i + V_i$ is not almost surely equal to 0 and denote by F its distribution function. We interpret U_i and V_i as alternating periods when a given machine is operational or in repair. The period U_1 begins at time 0. For $t \geq 0$ we define $Y_t := 1$ if the machine is operational at time t and $Y_t := 0$ otherwise. Let $g(t) := P[Y_t = 1]$ denote the probability of the machine being operational at time t.

a) *(Proof of Proposition 2.11)* Prove that g satisfies the renewal equation

$$g(t) = P[U_1 > t] + \int_0^t g(t-s)\, \mathrm{d}F(s), \quad t \geq 0.$$

b) Suppose further that $U_i \sim \mathrm{Exp}(\lambda)$ and $V_i \sim \mathrm{Exp}(\mu)$ for $i \in \mathbb{N}$ and $\lambda, \mu > 0$. Show that in this case g satisfies

$$g(t) = h(t) + \int_0^t g(t-s) f(s)\, \mathrm{d}s, \quad t \geq 0,$$

where $h(t) = \mathrm{e}^{-\lambda t}\mathbb{1}_{\{t\geq 0\}}$ and $f(t) = \frac{\lambda\mu}{\mu-\lambda}(\mathrm{e}^{-\lambda t} - \mathrm{e}^{-\mu t})\mathbb{1}_{\{t\geq 0\}}$.

Exercise 3-3

Vehicles of random lengths arrive at a gate. Let L_k denote the length of the kth vehicle. We assume that the random variables L_k are i.i.d. with $E[L_k] < \infty$. The first vehicle that arrives parks directly at the gate. The vehicles arriving afterwards queue behind, leaving a random distance to the vehicle parked in front of their own. We assume that these distances are independent and uniformly distributed on $[0, 1]$. For $x \geq 0$, let N_x denote the number of vehicles parked in distance at most x from the gate. Compute $\lim\limits_{x\to\infty} N_x/x$.

Exercise 3-4

Let $(N_t)_{t\geq 0}$ be a renewal process with i.i.d. interarrival times $(T_i)_{i\in\mathbb{N}}$ and let $S_n = \sum_{i=1}^n T_i$.

a) Assume that S_{N_t+1} is integrable for all $t \geq 0$ and let $A(t) = E[S_{N_t+1}]$. Show that $A(t)$ satisfies the renewal equation

$$A(t) = E[T_1] + \int_0^t A(t-x)\,\mathrm{d}F(x),$$

where F is the distribution function of T_1.

b) Show that for T_i bounded, S_{N_t+1} is integrable for all $t \geq 0$. Use the renewal equation derived above to prove that

$$E[S_{N_t+1}] = E[T_1](1 + M(t)), \qquad (*)$$

where $M(t)$ is the renewal function of $(N_t)_{t\geq 0}$.

c) Prove $(*)$ for general i.i.d. T_i with $P[T_i = 0] < 1$, $i \in \mathbb{N}$. *Hint:* use Fatou's lemma.

Exercise 3-5

Suppose the coffee machine of "Polysnack" has an average lifetime of two years and we assume that the lifetime distribution F admits a density. Upon failure the coffee machine has to be replaced immediately at a cost of 5000 CHF. Denote by M the renewal function corresponding to F. Assume that a new coffee machine has been bought today.

a) Show that the long-run average expenses per year for coffee machines are 2500 CHF.

"Polysnack" receives the offer to enter a long-term contract, which specifies that "Polysnack" has to buy coffee machines at a price of 3000 CHF at **each** of the predefined times T, $2T$, $3T$, $\ldots$, where $T > 0$ (in years) can be chosen. At other times the price of a coffee machine is still 5000 CHF.

b) Suppose that "Polysnack" enters the above contract. Then it has to replace machines at times T, $2T$, $3T$, $\ldots$, and upon failure. Show that the average expenses per year for coffee machines following the new policy are $(3000 + 5000M(T))/T$ CHF.

c) Give an example of a distribution F for which the new policy is more expensive than the old one for all $T > 0$.

d) Give an example of a distribution F and of a time T for which the new policy is at least 500 CHF cheaper per year than the old one.