ETH Zürich FS 2013
D-MATH
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Applied Stochastic Processes

Hints for Sheet 5

Exercise 5-1

Let Y_n denote the number which shows up in the n-th roll.

a) Consider two principally different outcomes of a roll.

b) Use $X_n = \max\{X_{n-1}, Y_n\}$. Both for the 1-step transition probability $r_{i,j}$ and the n-step transition probability $r_{i,j}(n)$, consider cases $j < i$, $j = i$ and $j > i$ separately.

c) Demonstrate a contradiction with the definition.

Exercise 5-2

a) Write down the matrix explicitly.

b) Derive a general formula for transition probabilities, but consider the cases $X_n = 0$ and $X_n = N$ separately.

c) As hinted in the exercise sheet, we can enumerate the states by $\{0, \ldots, 2^k - 1\}$. Express X_{n+1} from Y_{n+k+1} and X_n, using the remainder (modulo) operation $\mod 2^k$.

Exercise 5-3

Consider the possible outcomes after one step, starting from A, to obtain a linear equation for the hitting probabilities $\rho_{A,D}$, $\rho_{C,D}$ and $\rho_{E,D}$. Similarly, obtain linear equations involving $\rho_{A,E}$ and $\rho_{A,F}$. Reduce the number of variables using symmetry and solve this system of linear equations.

Exercise 5-4

Let $X := (X_n)_{n\in\mathbb{N}_0}$ be a homogeneous Markov chain with countable state space E. Without loss of generality we can assume that X is a *canonical* Markov chain with corresponding (raw) filtration $\mathbb{F} = (\mathcal{F}_n)_{n\in\mathbb{N}_0}$. The advantage of passing to the canonical process lies in the fact that we can use the *shift operator* θ to describe the Markov property (see lecture notes).

Recall that for $x, y \in E$ and $n \in \mathbb{N}$ we defined the quantities

$$r_{x,y}(n) := P_x[X_n = y], \quad r_{x,y} := r_{x,y}(1),$$

the probability of being in state y after n steps when started at x. One can correspondingly define the transition probability for a *set* $C \subset E$ as

$$r_{x,C}(n) := P_x[X_n \in C] \left(= \sum_{y\in C} P_x[X_n = y] = \sum_{y\in C} r_{x,y}(n) \right). \tag{1}$$

Instead of looking at hitting probabilities, one can ask for the *time* X needs to hit C. Therefore, we define

$$\tau_C := \inf\{n \geq 0;\, X_n \in C\}. \tag{2}$$

The relation between transition probabilities and τ_C is given by the next lemma.

Lemma 1. The following are equivalent:

(i) $P_x[\tau_C < \infty] > 0$ for all $x \in E\backslash C$;

(ii) for each $x \in E\backslash C$ there exists $n(x) \in \mathbb{N}$ such that $r_{x,C}(n(x)) > 0$.

Hint for the proof:
"(ii) ⇒ (i)": Use $\{\tau_C < \infty\} = \bigcup_n \{X_n \in C\}$.
"(i) ⇒ (ii)": Since

$$0 < P_x[\tau_C < \infty] = \sum_n P_x[\tau_C = n],$$

we conclude that there exists $n(x)$ such that $P_x[\tau_C = n(x)] > 0$. Find an expression of the set $\{\tau_C = n(x)\}$ in terms of the X_k's and deduce (ii).

Now, in order to show

$$P_x[\tau_C > kN] \leq (1-\varepsilon)^k \quad \forall x \in E, \tag{3}$$

it is reasonable to exploit the Markov property of X which is hidden in the quantity τ_C.

Prove the following lemma.

Lemma 2.

(i) $\{\tau_C > kN\} = \{X_0 \in E\backslash C, X_1 \in E\backslash C, \ldots, X_{kN} \in E\backslash C\}$,

and hence,

(ii) $\mathbb{1}_{\{\tau_C > kN\}} = \mathbb{1}_{\{\tau_C > (k-1)N\}} \left(\mathbb{1}_{\{\tau_C > N\}} \circ \theta_{(k-1)N} \right)$.

Due to Lemma 2 (ii) and the Markov property, an induction argument on k for the proof of (3) seems reasonable.

Hint: Show (3) for $k = 1$ and use Lemma 2 (ii) and the Markov property for the induction step.

Exercise 5-5

a) Define for $x \in E$ the (bounded Borel) function $h(x) := P_x[\tau_A < \tau_B]$, where the sets fulfill the assumptions of the exercise. The function h denotes the probability that X hits A before B when started at x.

   Show the relation
   $$h(x) = \sum_{y \in E} r_{x,y} h(y) \quad \forall x \in E \backslash (A \cup B). \tag{4}$$

   Hint: Use the Markov property to show that
   $$\sum_{y \in E} r_{x,y} h(y) = E_x\left[\mathbb{1}_{\{\tau_A < \tau_B\}} \circ \theta_1\right].$$

   Since we would like to show that this is equal to $h(x)$, it is natural to guess that $\mathbb{1}_{\{\tau_A < \tau_B\}} = \mathbb{1}_{\{\tau_A < \tau_B\}} \circ \theta_1$ P_x-a.s. Show that this guess is indeed correct by using the decomposition
   $$\{\tau_A < \tau_B\} = \bigcup_n \{\tau_A = n, \tau_B > n\}.$$

   Conclude (4).

b) Lemma 1 implies that the assumptions of Exercise 5-4 are satisfied, and consequently
   $$P_x[\tau_{A \cup B} > kN] \leq (1 - \varepsilon)^k.$$

   Conclude that $\tau_{A \cup B} < \infty$ P_x-a.s.