Homework 5

1. Let Ω be a smooth domain. Let $f \in L^2(\Omega)$ and $u \in H_0^2(\Omega)$ (which is the space of H^2-functions on Ω whose derivatives are in $H_0^1(\Omega)$) satisfy

$$\Delta^2 u = f,$$

where $\Delta^2 u = \Delta(\Delta u)$. Define

$$\langle u, v\rangle_* := \int_\Omega \Delta u \Delta v.$$

a) Find a weak formulation of the boundary value problem

$$\begin{cases} \Delta^2 u = f, \\ 0 = u = \partial_\nu u = 0 \text{ on } \partial\Omega, \end{cases}$$

using $\langle \cdot, \cdot \rangle_*$.

b) Show that for any $f \in L^2(\Omega)$ there exists a unique $u \in H_0^2(\Omega)$ such that

$$\langle u, v\rangle_* = \langle f, v\rangle_{L^2}$$

holds for all $v \in H_0^2(\Omega)$.

c) Show that the weak solution satisfies

$$\|u\|_{H^4(\Omega)} \leq C\|f\|_{L^2(\Omega)}$$

where C is a constant independent of f.

2. Let Ω be a smooth domain with compact boundary, $f \in H^{-1}(\Omega) = [H_0^1(\Omega)]^*$, $g \in H^{1/2}(\partial\Omega)$. Prove that the boundary value problem

$$\begin{cases} -\Delta u &= f \\ u|_{\partial\Omega} &= g \end{cases}$$

has a unique weak solution u, which satisfies also the estimate

$$||u||_{H^1} \leq C(||f||_{H^{-1}} + ||g||_{H^{1/2}}).$$

Hint: First prove the case $g = 0$.

3. Consider the boundary value problem

$$\begin{cases} -\Delta u &= \lambda u + f \\ u|_{\partial\Omega} &= 0 \end{cases}$$

for $f \in H^{-1}(\Omega)$. We look for a solution $u \in H_0^1(\Omega)$. Recall that $\langle u, v\rangle_{H_0^1} := \langle \nabla u, \nabla v\rangle_{L^2}$ is a nondegenerate inner product on $H_0^1(\Omega)$.

Please turn over!

a) Prove that there exists a continuous linear operator $A : H_0^1(\Omega) \to H_0^1(\Omega)$ such that for all $u, \varphi \in H_0^1(\Omega)$,

$$\int_\Omega u\varphi = \langle Au, \varphi \rangle_{H_0^1}.$$

b) Prove that the above operator A is compact and positive definite.

c) Prove that if $f \in H^{-1}(\Omega)$ then there is $v \in H_0^1(\Omega)$ such that $\int_\Omega f\varphi = \langle v, \varphi \rangle_{H_0^1}$ for all $\varphi \in H_0^1(\Omega)$.

d) Show that the following stronger version of the boundary value problem

$$\forall \varphi \in H_0^1(\Omega), \quad \langle u - \lambda Au, \varphi \rangle_{H_0^1} = \langle v, \varphi \rangle_{H_0^1},$$

has a unique solution u satisfying $||u||_{H^1} \leq ||f||_{H^{-1}}$ if $\lambda \notin \text{spec}A$. Recall that

$$\text{spec}A = \{\lambda \in \mathbb{R} : (Id - \lambda A) : H_0^1(\Omega) \to H_0^1(\Omega) \text{ does not have a bounded inverse}\}.$$

4. Let Ω be an open domain with smooth boundary. Assume $u \in H^1(\Omega)$ is a bounded weak solution of

$$-\sum_{i,j=1}^{n} (a^{i,j} u_{x_i})_{x_j} = 0 \qquad \text{in } \Omega,$$

where the coefficients of this PDE are smooth and satisfy the uniform ellipticity condition. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be convex and smooth, and set $w = \varphi(u)$. Show w is a weak subsolution; that is,

$$B[w, v] \leq 0$$

for all $v \in H_0^1(\Omega), v \geq 0$.

Due: May 6, 2013.