Homework 6

In the following exercises, assume $\Omega \subset \mathbb{R}^n$ is an open bounded set with smooth boundary.

1. Let $\{w_k\}$ be both an orthogonal basis of H_0^1 and an orthonormal basis of L^2. Suppose $f \in L^2(\Omega)$ and assume that $u_m = \sum_{k=1}^m d_m^k w_k$ solves

$$\int_\Omega Du_m \cdot Dw_k \, dx = \int_\Omega f \cdot w_k \, dx$$

for $k = 1, \dots, m$. Show that a subsequence of $\{u_m\}_{m=1}^\infty$ converges weakly in $H_0^1(\Omega)$ to the weak solution u of

$$\begin{cases} -\Delta u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{cases}.$$

2. Suppose that H is a Hilbert space and

$$\mathbf{u}_k \rightharpoonup \mathbf{u} \qquad \text{in } L^2(0,T;H),$$

where $T > 0$ and $\rightharpoonup$ denotes weak convergence in $L^2(0,T;H)$, the space of strongly measurable functions $\mathbf{f} : (0,T) \to H$ satisfying

$$\left(\int_0^T \|\mathbf{f}(t)\|_H^2 \, dt \right)^{\frac{1}{2}} < \infty.$$

(See Section 5.9.2 and Appendix E.5 of Evans.) Suppose further we have the uniform bounds

$$\operatorname{ess\,sup}_{0 \le t \le T} \|\mathbf{u}(t)\|_H \le C \quad (k = 1, \dots),$$

for some constant C. Prove

$$\operatorname{ess\,sup}_{0 \le t \le T} \|\mathbf{u}(t)\|_H \le C.$$

Hint: If $v \in H$ and $0 \le a \le b \le T$, we have

$$\int_a^b (v, \mathbf{u}_k(t)) \, dt \le C\|v\|_H |b-a|.$$

3. Let u be a smooth solution of $Lu = -\sum_{i,j=1}^n a^{ij} u_{x_i x_j} = 0$ in Ω, where L is uniformly elliptic with smooth coefficients. Set $v = |Du|^2 + \lambda u^2$. Show that

$$Lv \le 0 \quad \text{in } \Omega, \text{ if } \lambda \text{ is large enough.}$$

Deduce

$$\|Du\|_{L^\infty(\Omega)} \le C(\|Du\|_{L^\infty(\partial\Omega)} + \|u\|_{L^\infty(\partial\Omega)}).$$

Please turn over!

4. Assume u is a smooth solution of $Lu = -\sum_{i,j=1}^{n} a^{ij} u_{x_i x_j} = f$ in Ω, $u = 0$ on $\partial\Omega$, where f is bounded. (Again, we assume that L is uniformly elliptic with smooth coefficients.) Fix $x^0 \in \partial\Omega$. A *barrier* at x^0 is a C^2-function w such that

$$Lw \geq 1 \text{ in } \Omega, \quad w(x^0) = 0, \quad w \geq 0 \text{ on } \partial\Omega.$$

Show that if w is a barrier at x^0, there exists a constant C such that

$$|Du(x^0)| \leq C \left| \frac{\partial w}{\partial \nu}(x^0) \right|.$$

Due: May 27, 2013.