Selected Solutions 1

1.

2.

3. First, some notation. Consider the "cross" formed by the lines $x_1 = x_2$ and $x_1 = -x_2$. Let A be the triangle formed by the cross along with the line $x_1 = 1$, B the triangle formed by the cross along with the line $x_2 = 1$, C the triangle formed by the cross along with the line $x_1 = -1$, and finally D the triangle formed by the cross along with $x_2 = -1$.

Since $u(x)$ is a bounded continuous function on the bounded set U, certainly $u \in L^p$ for all $1 \leq p \leq \infty$. What about the weak derivatives? We claim that the weak derivatives of u are

$$v_1(x) = \begin{cases} -1 & x \in A \\ 1 & x \in C \\ 0 & x \in B \cup D \end{cases}$$

and

$$v_2(x) = \begin{cases} -1 & x \in B \\ 1 & x \in D \\ 0 & x \in A \cup C \end{cases}.$$

If the claims holds, then since these are simple functions, they certainly are in L^p for all p considered.

Verifying that these functions are indeed the weak derivatives is an exercise in the use of higher dimensional integration by parts. We provide a sketch for v_1. (The other case is similar.)

Let $\varphi \in C^\infty(U)$. We wish to show that

$$\int u\varphi_{x_1}\,dx = -\int v\varphi\,dx \qquad \left(= \int_A \varphi\,dx - \int_C \varphi\,dx\right). \tag{1}$$

We will integrate $u\varphi_{x_1}$ triangle by triangle where the strong derivatives of u are well-defined, starting with A:

$$\int_A u\varphi_{x_1}\,dx = -\int_A u_{x_1}\varphi\,dx + \int_{\partial A} u\varphi\nu^1 dS,$$

where $\nu(x) = (\nu^1(x), \nu^2(x))$ is the outward pointing unit normal vector. Since A is a triangle, we compute that

$$\nu(x) = \begin{cases} (1,0) & x \in A \cap \{x_1 = 1\} \\ (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) & x \in A \cap \{x_1 = x_2\} \\ (-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) & x \in A \cap \{x_1 = -x_2\} \end{cases}.$$

We also note that on A, $u_{x_1} = -1$ (in the strong sense). Thus, we have

$$\int_A u\varphi_{x_1}\,dx = \int_A \varphi\,dx + 0 + \int_{A\cap\{x_1=x_2\}} -\frac{1}{\sqrt{2}}(1-x_1)\varphi\,dS + \int_{A\cap\{x_1=-x_2\}} -\frac{1}{\sqrt{2}}(1-x_1)\varphi\,dS.$$

Please turn over!

The first term of the sum matches with one term of the sum in parentheses in (1). The other two terms cancel with some of the terms appearing in the expansions (in terms of integration by parts) of the integrals $\int_B u\varphi_{x_1}\,dx$ and $\int_D u\varphi_{x_1}\,dx$. In fact, the desired cancellation occurs for the integrals over all triangles showing that Equation (1) is valid.

4. First we must show that $u(x) \in L^n(U)$. Passing to (hyper)spherical coordinates, we have

$$\int_U |u(x)|^n\,dx \ll \int_0^1 r^{n-1}|\log\log(1+\frac{1}{r})|\,dr.$$

The integrand is continuous everywhere in U except perhaps at $r = 0$. However, L'Hopital's rule tells us that

$$\lim_{r\to 0} r^{n-1}|\log\log(1+\frac{1}{r})| = 0.$$

Hence the integrand is continuous, and the integral is finite.

Note that u is smooth in U away from the origin $x = 0$. In fact,

$$\partial_{x_j} u \;=\; -\,x_j\,|x|^{-2}\left(1+|x|\right)^{-1}\left[\log\left(1+|x|^{-1}\right)\right]^{-1} \qquad \forall\; x\neq 0\,.$$

Hence,

$$|Du| \;=\; |x|^{-1}\left(1+|x|\right)^{-1}\left[\log\left(1+|x|^{-1}\right)\right]^{-1} \qquad \forall\; x\neq 0\,.$$

Away from the origin, this function belongs to L^n. We are thus only interested in finding the integrability of $|Du|$ near $x = 0$. There, we see that

$$|Du| \;\sim\; |x|^{-1}\left[\log\left(|x|^{-1}\right)\right]^{-1} \qquad \forall\; x\neq 0\,.$$

Thus, passing to (hyper)spherical coordinates (with $r = |x|$) yields

$$\int_U |Du|^n\,dx \;\sim\; \int_0^\epsilon r^{-1}\left(\log\frac{1}{r}\right)^{-n} dr \;+\; C_\epsilon\,,$$

for all $\epsilon \in (0,1)$ and some finite constant C_ϵ.

Note that

$$\int_0^\epsilon r^{-1}\left(\log\frac{1}{r}\right)^{-n} dr \;=\; \int_{-\log\epsilon}^\infty s^{-n}\,ds \;<\; \infty.$$

Using these estimates along with the Dominated Convergence Theorem, we are able to conclude that for any $\varphi \in C^\infty(U)$

$$\int_U u\varphi_{x_i}\,dx = -\int_U u_{x_i}\varphi\,dx$$

and hence u belongs to $W^{1,n}(U)$.

5.