Solutions 5

1. **a)** First we should verify that $\langle u, v\rangle_*$ is a scalar product. The only property which is not immediately clear is that $u = 0$ only if $\langle u, u\rangle_* = 0$. To see this, note that $\langle u, u\rangle_* = ||\Delta u||^2_{L^2}$, so if this is zero then u is a H^2_0-solution of

$$\begin{cases} \Delta u = 0 \\ u|_{\partial\Omega} = 0 \end{cases}$$

But we know that there exists a unique weak solution of this problem in $H^1_0(\Omega)$, and a possible solution is the zero function. Therefore, $u = 0$.

The weak formulation of the equation is

$$\langle u, v\rangle_* = \langle f, v\rangle_{L^2}, \quad \text{for all test functions } v.$$

This can be proved by applying integrations by parts twice, using the fact that $u = \partial_\nu u = 0$ on the boundary.

b) Define on $H^2_0(\Omega)$ the linear functional $L_f : v \mapsto \langle f, v, \rangle_{L^2}$. This functional is continuous with respect to the H^2-norm, but to apply the Riesz Representation Theorem in a meaningful way we need continuity with respect to $\langle\cdot,\cdot\rangle_*$. Clearly $||\nabla^2 u||_{L^2} \geq ||\Delta u||_{L^2}$. From the estimates of Sections 9.3 and 9.4, we have the inequality $||u||_{H^2} \leq C||u||_*$, thus demonstrating equivalence of the two norms in question. Therefore, by Riesz, there exists a unique funtion $u \in H^2_0(\Omega)$ which satisfies the weak equation for all $v \in H^2_0(\Omega)$.

c) Applying Theorem 5 of Chapter 6.3 of Evans, we have that

$$\|u\|_{H^4} \leq C_k \|\Delta u\|_{H^2}.$$

We also have assumed that $\Delta^2 u = f$ weakly, that is, Δu weakly solves

$$\Delta(\Delta u) = f.$$

Applying another version of elliptic regularity (this assumes that Δu is in H^1_0, which at this point is unknown), we have

$$\|\Delta u\|_{H^2} \leq C_0 \|\Delta\Delta u\|_{L^2} = C_0 \|f\|_{L^2}.$$

Combing these two inequalities yields the desired result.

To have a full regularity statement (rather than just a statement about interior regularity), one must proceed as in the proofs of the theorems regarding elliptic regularity with the usual tricks of extension to the whole space and flattening the boundary.

2. We first consider the case $g = 0$. The scalar product

$$\langle u, v\rangle_* = \int_\Omega \nabla u \cdot \nabla v$$

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is equivalent to the standard one on $H_0^1(\Omega)$, due to the Poincaré inequality.
The functional $L_f(\varphi) = \int_\Omega f\varphi$ helps to write the weak equation in the form

$$\langle v, \varphi\rangle_* = L_f(\varphi), \forall\varphi.$$

By Riesz Theorem the continuous functional L_f on $H_0^1(\Omega)$ has a unique representative $v \in H_0^1(\Omega)$, and has the same norm as the norm of v:

$$||L_f|| := \sup\left\{|L_f(\varphi)|/\langle\varphi,\varphi\rangle_*^{1/2} : \varphi \in H_0^1(\Omega)\setminus\{0\}\right\} = \langle v, v\rangle_*^{1/2}.$$

In other words v is the unique weak solution of our equation, and we have an estimate on its norm w.r.t. $\langle\cdot,\cdot\rangle_*$.
We know that the following is finite:

$$||f||_{H^{-1}} = \sup\left\{|L_f(\varphi)|/||\varphi||_{H^1} : \varphi \in H_0^1(\Omega)\setminus\{0\}\right\}.$$

Because the two norms on $H_0^1(\Omega)$ are equivalent, we obtain

$$C^{-1}||f||_{H^{-1}} \leq ||L_f|| \leq C||f||_{H^{-1}}$$

therefore we have the estimate

$$||v||_{H^1} \leq C||f||_{H^{-1}},$$

as desired

We now pass to the case $g \neq 0$. By the extension theorem for $g \in H^{1/2}(\partial\Omega)$ we know that there exists $G \in H^1(\Omega)$ with trace on the boundary g and

$$||G||_{H^1(\Omega)} \leq C||g||_{H^{1/2}(\partial\Omega)}.$$

Suppose v solves

$$\begin{cases} -\Delta u = f \\ u|_{\partial\Omega} = g. \end{cases}$$

Then $\hat{v} = v - G$ solves

$$\begin{cases} -\Delta u = f + \Delta G \\ u|_{\partial\Omega} = 0. \end{cases}$$

By the above discussion, $\hat{v}$ is the unique solution of this system and satisfies

$$\|\hat{v}\|_{H^1} \leq C\|f + \Delta G\||_{H^{-1}}$$

since $f + \Delta G \in H^{-1}(\Omega)$. Thus, using the fact that Δ is a bounded operator, we have

$$\|v\| - \|G\| \leq \|\hat{v}\| \ll \|f + \Delta G\| \ll \|f\| + \|G\|$$

yielding

$$\|v\| \ll \|f\| + \|g\|.$$

3. **a)** We have that $\langle u, v\rangle := \int \nabla u \cdot \nabla v$ is a nondegenerate scalar product on $H_0^1(\Omega)$, inducing a norm equivalent to the standard H^1-norm. The bilinear form $u, v \mapsto \int uv$ is continuous on $H_0^1(\Omega)$, so by Riesz theorem it can be represented as $\langle Au, v\rangle$, where A is a continuous linear operator.

See next page!

b) Clearly A is self-adjoint, and also $\langle Au, u\rangle = ||u||_{L^2(\Omega)}$ so it is also a positive operator.
To prove compactness of A we will use the fact that an operator on a Hilbert space is compact if and only if whenever a sequence u_k converges weakly, the sequence Au_k converges strongly. Let u_k be a sequence converging weakly in $H_0^1(\Omega)$. In particular the norms $||u_k||_{H^1(\Omega)}$ are bounded, therefore $||Au_k||_{H^1(\Omega)}$ are also uniformly bounded by the boundedness of A. Because the injection $H^1(\Omega) \to L^2(\Omega)$ is compact, it follows that u_k converge strongly in L^2. We will now use all these facts and the following computation:

$$\begin{aligned}\langle A(u_k - u_l), A(u_k - u_l)\rangle &= \int_\Omega (u_k - u_l)(Au_k - Au_l) \\ &\leq ||u_k - u_l||_{L^2}||Au_k - Au_l||_{L^2}.\end{aligned}$$

In the last term above we have that the first factor converges to 0, while the second factor is bounded. Therefore the whole product converges to 0. Since the H^1-norm is equivalent with the one induced by our scalar product, we just proved that Au_k is converging strongly, as desired.

c) Here the proof is the same as in exercise 2, via Riesz representation theorem.

d) By reflexivity of the Hilbert space $H_0^1(\Omega)$, the weak formulation is equivalent to the following strong one:

$$u - \lambda Au = v.$$

If $\lambda \notin \operatorname{spec} A$ then for all $v \in H_0^1(\Omega)$ we find an unique u satisfying the above equation. Furthermore, we have

$$||(Id - \lambda A)^{-1}(v)||_{H^1} \leq C_\lambda ||v||_{H^1}$$

and since $||v||_{H^1} = ||f||_{H^{-1}}$, we have that $u = (Id - \lambda A)^{-1}(v)$ satisfies the norm estimate.

4. An outline of the calculation is as follows: Apply integration by parts once to write B in terms of φ'', and obtain the desired inequality using the elliptic PDE and the fact that convexity implies positivity of φ''.