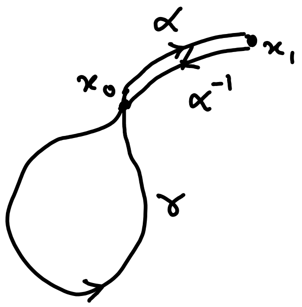
Let X be a topological space and $x_0 \in X$.
Recall that $\pi_1(X, x_0) := {}_{x_0}X_{x_0}/_\sim$ has a group structure given by composition of paths.

Let $x_1 \in X$ and assume that there exists a path $\alpha \in {}_{x_0}X_{x_1}$.
We define a map $\hat{\alpha}: \pi_1(X,x_0) \to \pi_1(X,x_1)$, $[\gamma] \mapsto [\alpha]^{-1} * [\gamma] * [\alpha]$.



Claim: $\hat{\alpha}$ is a group isomorphism.

Proof: we first prove that it is a group homomorphism:

$$\hat{\alpha}(\underbrace{[\gamma_1]*[\gamma_2]}_{\substack{\| \\ [\gamma_1 * \gamma_2]}}) = [\alpha]^{-1} * [\gamma_1] * [\gamma_2] * [\alpha]$$
$$= [\alpha]^{-1} * [\gamma_1] * [\alpha] * [\alpha]^{-1} * [\gamma_2] * [\alpha]$$
$$= \hat{\alpha}([\gamma_1]) * \hat{\alpha}([\gamma_2])$$

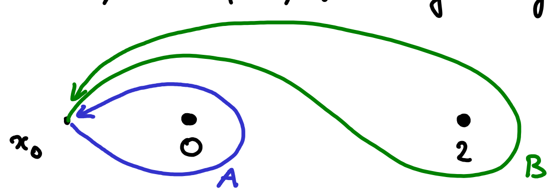
Note that $\widehat{\alpha^{-1}} \circ \hat{\alpha} = \hat{\alpha} \circ \widehat{\alpha^{-1}} = id$. Hence $\hat{\alpha}$ is an isomorphism. □.

Corollary: if X is path-connected then $\pi_1(X,x_0)$ and $\pi_1(X,x_1)$ are isomorphic for any two points $x_0, x_1 \in X$.

⚠ the identification between $\pi_1(X,x_0)$ and $\pi_1(X,x_1)$ is NOT canonical. Namely, $\hat{\alpha}$ depends on $[\alpha]$.

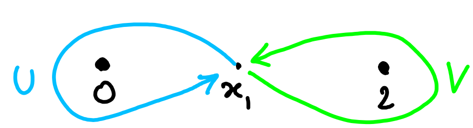
Examples: $X = \mathbb{C} - \{0, 2\}$, $x_0 = -1$ and $x_1 = 1$

Let $A, B \in \pi_1(X, x_0)$ be given by the following loops:

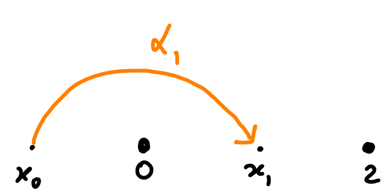


Actually, $\pi_1(X,x_0)$ is the free group generated by A and B (we will see this later).

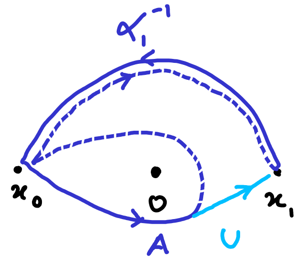
Let $U, V \in \pi_1(X, x_1)$ be given by the following loops:



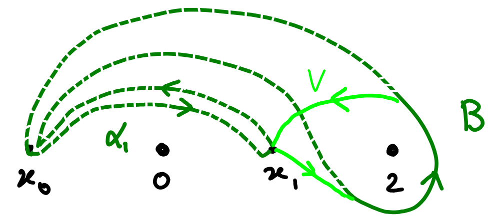
Let now $\alpha_1 \in {}_{x_0}X_{x_1}$ be as follows:



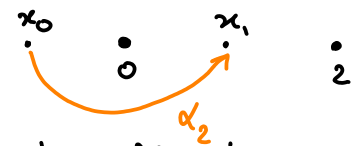
One can show that $\hat{\alpha}_1(A) = U$ and $\hat{\alpha}_1(B) = V$. Namely:



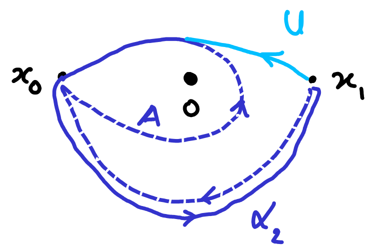
and



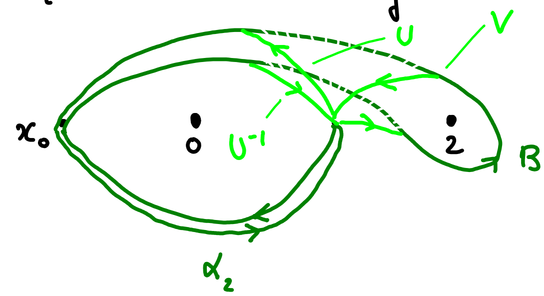
Let finally $\alpha_2 \in {}_{x_0}X_{x_1}$ be as follows:



One can show that $\hat{\alpha}_2(A) = U$ and $\hat{\alpha}_2(B) = U^{-1}VU$. Namely:



and



More generally, if $\alpha \in {}_{x_0}X_{x_1}$ and $\beta \in {}_{x_1}X_{x_2}$ then $\hat{\beta} \circ \hat{\alpha} = \widehat{\alpha * \beta}$ (exercise).
In particular if $\alpha_1, \alpha_2 \in {}_{x_0}X_{x_1}$ then $\hat{\alpha}_2 : \pi_1(X, x_0) \to \pi_1(X, x_1)$ is obtain from $\hat{\alpha}_1$ by conjugation with $[\alpha_1 * \alpha_2^{-1}] \in \pi_1(X, x_1)$.

<u>Definition</u>: X is <u>simply connected</u> if it is path-connected and $\pi_1(X, x_0) = \{[e_{x_0}]\}$ for some $x_0 \in X$ (and thus for all of them).

Note that if X is simply connected then for any $\alpha, \beta \in {}_{x_0}X_{x_1}$, $\alpha \sim \beta$.
Namely, $\alpha \sim \beta \Leftrightarrow \alpha * \beta^{-1} \sim e_{x_0} \Leftrightarrow [\alpha * \beta^{-1}] = [e_{x_0}] \in \pi_1(X, x_0)$.

<u>Definition</u>: let $f : X \to Y$ be a continuous map with $y_i = f(x_i)$, $i = 0$ or 1.
For any $\gamma \in {}_{x_0}X_{x_1}$ we define $f_* \gamma := f \circ \gamma \in {}_{y_0}Y_{y_1}$.

<u>Properties</u>: 1) if $\eta \in {}_{x_1}X_{x_2}$ then $f_*(\gamma * \eta) = (f_* \gamma) * (f_* \eta)$.
2) if $\gamma' \sim \gamma$ then $f_* \gamma' \sim f_* \gamma$.

This implies in particular that f_* induces a group homomorphism $\pi_1(X, x_0) \to \pi_1(Y, y_0)$.

<u>Proof of the properties</u>: • the first one is obvious:

$$f_*(\gamma * \eta) = f \circ (\gamma * \eta) = (f \circ \gamma) * (f \circ \eta) = (f_* \gamma) * (f_* \eta).$$

• the second is not that much harder:
if H is a homotopy between γ and γ' then $f \circ H$ is a homotopy between $f_* \gamma = f \circ \gamma$ and $f_* \gamma' = f \circ \gamma'$. □

If we have continuous maps $f: X \to Y$ and $g: Y \to Z$ with $y_0 = f(x_0)$ and $z_0 = g(y_0)$ then $(g\circ f)_* = g_* \circ f_*$ [Proof: $(g\circ f)_*(\gamma) = (g\circ f)\circ\gamma = g\circ(f\circ\gamma) = g_*(f_*\gamma)$. $\square$]

Consequence: if $f: X \to Y$ is a homeomorphism with $y_0 = f(x_0)$, then f_* induces an isomorphism $\pi_1(X, x_0) \to \pi_1(Y, y_0)$.

Proof: let g be the inverse of f. Then $g_*: \pi_1(Y, y_0) \to \pi_1(X, x_0)$ is the inverse of f_* (it is obvious that $(id_X)_* = id_{\pi_1(X,x_0)}$). $\square$

Proposition: $\pi_1(X, x_0)$ acts freely and transitively on ${}_{x_0}X_{x_1}/_\sim$ (via $\gamma * \alpha$).

Recall that if G is a group acting (from the left) on a set A then we say that the action is:

- free if there are no stabilizers: $g\cdot a = a \Rightarrow g = e$.
- transitive if there is only one orbit: $\forall a, b \in A, \exists g \in G$ s.t. $b = g\cdot a$.

Note that "free + transitive" $\Leftrightarrow \forall a, b \in A, \exists! g \in G$ s.t. $b = g\cdot a$.

Proof of the proposition: the action is
$$\pi_1(X, x_0) \times ({}_{x_0}X_{x_1}/_\sim) \to ({}_{x_0}X_{x_1}/_\sim)$$
$$([\gamma], [\alpha]) \longmapsto [\gamma] * [\alpha] = [\gamma * \alpha].$$

- For any two $\alpha, \beta \in {}_{x_0}X_{x_1}$ we define $\gamma = \beta * \alpha^{-1}$.
  Then $[\gamma] * [\alpha] = [(\beta * \alpha^{-1}) * \alpha] = [\beta * (\alpha^{-1} * \alpha)] = [\beta]$. The action is transitive.
- Let $\alpha \in {}_{x_0}X_{x_1}$ and $\gamma \in {}_{x_0}X_{x_0}$ be such that $[\gamma * \alpha] = [\alpha]$.
  Hence $[\gamma] = [\gamma * (\alpha * \alpha^{-1})] = [(\gamma * \alpha) * \alpha^{-1}] = [\gamma * \alpha] * [\alpha]^{-1} = [\alpha] * [\alpha]^{-1} = [e_{x_0}]$.

The action is free. $\square$.

In particular, if ${}_{x_0}X_{x_1} \neq \emptyset$ then any $\alpha \in {}_{x_0}X_{x_1}$ provides a bijection
$$\pi_1(X, x_0) \to ({}_{x_0}X_{x_1}/_\sim)\ ;\ [\gamma] \mapsto [\gamma * \alpha].$$