2) Covering spaces (a tool for computing fundamental groups)

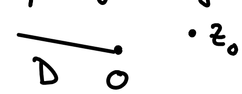
Definition: a surjective continuous map $p : E \to B$ is a <u>covering map</u> if for any $b \in B$ there exists a neighbourhood U of b, a set S and a homeomorphism $\varphi : p^{-1}(U) \to U \times S = \coprod_{s \in S} U$ such that the following diagram commutes:

$$\begin{tikzcd} p^{-1}(U) \arrow[r, "\varphi"] \arrow[dr, "p"'] & U \times S \arrow[d, "\leftarrow \text{first projection}"] \\ & U \end{tikzcd}$$

E is then called a <u>covering space</u> of B and U is called a <u>trivializing open</u> w.r.t. p.

Examples:

- the first projection $E := B \times S \to B$ is a covering map. It is called the <u>trivial covering</u>.
- $E = \mathbb{R} \longrightarrow S^1 = \mathbb{R}/\mathbb{Z} = B$ is a covering map. Namely, for any $b \in \mathbb{R}/\mathbb{Z}$ we choose a representative $e \in \mathbb{R}$ of b and consider $U := p\left(\left(e - \frac{1}{2}, e + \frac{1}{2}\right)\right)$.
  Then $p^{-1}(U) = \coprod_{k \in \mathbb{Z}} \left(e + k - \frac{1}{2}, e + k + \frac{1}{2}\right) \xleftarrow[\varphi^{-1}]{\sim} U \times \mathbb{Z}$
  $$x + k \longleftarrow (p(x), k), \quad x \in \left(e - \tfrac{1}{2}, e + \tfrac{1}{2}\right)$$
- $E = \mathbb{R} \longrightarrow S^1 = \{ z \in \mathbb{C} \mid |z| = 1 \}$ is a covering map.
  $x \longmapsto e^{2\pi i x}$
  (This is because essentially the same example as before).
- $E = \mathbb{C} \longrightarrow \mathbb{C}^\times$; $\xi \mapsto e^{i\xi}$ is a covering map. Namely, for any $z_0 \in \mathbb{C}^\times$ we choose a half-line D starting at 0 and not passing through z_0.
  We let $U := \mathbb{C}^\times - D$.



  Then $p^{-1}(U) = [\xi \in \mathbb{C} \mid Re(\xi) \neq \arg(D) \bmod 2\pi \}$
  $$\varphi : p^{-1}(U) \longrightarrow U \times \mathbb{Z}$$
  $$\xi \longmapsto \left(e^{i\xi}, E\left[\frac{Re(\xi) - \arg(D)}{2\pi}\right]\right)$$
  "integer part"

  φ^{-1} sends (z, k) to $\frac{1}{i}\log(z)$, where the determination of the log that we choose is determined by the choice of D and of the integer k.
- $\mathbb{R}/\mathbb{Z} \longrightarrow \mathbb{R}/\mathbb{Z}$; $x \bmod 1 \mapsto 2x \bmod 1$ is a covering map (exercise).

Properties: 1) if $E \xrightarrow{p} B$ is a covering map and $B_0 \subset B$, then $p^{-1}(B_0) \xrightarrow{p} B_0$ is a covering map.

2) if $E_i \xrightarrow{p_i} B_i$ $(i=1,2)$ are covering maps, then $E_1 \times E_2 \xrightarrow{p_1\times p_2} B_1 \times B_2$ is a covering map.

Proof: 1) let $b \in B_0 \subset B$. There exists U neigh. of b in B, S set, and $\varphi: p^{-1}(U) \longrightarrow U\times S$ homeomorphism such that
$$\begin{tikzcd} p^{-1}(U) \arrow[r, "\varphi"] \arrow[dr, "p"'] & U\times S \arrow[d] \\ & U \end{tikzcd}$$
commutes.

In particular if we restrict ourselves to $U_0 := U \cap B_0$ (neigh. of b in B_0) we get a commuting diagram
$$\begin{tikzcd} p^{-1}(U_0) \arrow[r, "\varphi"] \arrow[dr, "p"'] & U_0\times S \arrow[d] \\ & U_0 \end{tikzcd}.$$

2) let $b=(b_1,b_2) \in B_1 \times B_2$. Then for $i=1$ or 2 we have U_i neigh. of b_i in B_i, S_i set and $\varphi_i : p_i^{-1}(U_i) \longrightarrow U_i \times S_i$ homeomorphism such that
$$\begin{tikzcd} p^{-1}(U_i) \arrow[r, "\varphi"] \arrow[dr, "p"'] & U_i\times S \arrow[d] \\ & U_i \end{tikzcd}$$
commutes. Then let $U := U_1 \times U_2$, $S = S_1 \times S_2$ and $\varphi = \varphi_1 \times \varphi_2$.

Then we have a commuting diagram:
$$\begin{tikzcd} (p_1\times p_2)^{-1}(U) \simeq p_1^{-1}(U_1)\times p_2^{-1}(U_2) \arrow[r, "\varphi_1\times\varphi_2"] \arrow[dr] & U_1\times S_1 \times U_2 \times S_2 \simeq U\times S \arrow[d] \\ & U_1 \times U_2 = U \end{tikzcd}$$
$\square$

Consequence: $\mathbb{R}\times\mathbb{R} \rightarrow S^1 \times S^1$ is a covering map.

The main interest of covering maps is that any loop in B can be lifted to a path in the covering space E, allowing to compute the fundamental group of B from the information carried by the (somehow easier) topology on E.

Definition: let $p: E \rightarrow B$ and $f: X \rightarrow B$ be continuous maps. A **lift of f** (along p) is a continuous map $\tilde{f}: X \rightarrow E$ such that $p\circ\tilde{f} = f$.

In terms of a commuting diagram $p\circ\tilde{f} = f$ reads
$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & B \end{tikzcd}.$$

Example: the continuous map $f: [0,1] \longrightarrow \mathbb{R}/_{\mathbb{Z}}$, $x \longmapsto x \bmod 1$ admits the following lifts along $p: \mathbb{R} \rightarrow \mathbb{R}/_{\mathbb{Z}}$: $\tilde{f}: [0,1] \longrightarrow \mathbb{R}$, $x \longmapsto x+k$ $(k \in \mathbb{Z})$.

The above example motivates the following:

Lemma: let $p: E \rightarrow B$ be a covering map and let $e_0 \in E$. Any path $\gamma: [0,1] \longrightarrow B$ starting at $b_0 = p(e_0)$ admits a unique lift $\tilde{\gamma}: [0,1] \rightarrow E$ starting at e_0.

Proof: we start with the following

Claim: there exists $0 = t_0 \leq t_1 \leq \cdots \leq t_n = 1$ a finite subdivision of $[0,1]$ such that $\gamma([t_i, t_{i+1}])$ lies in a trivializing open subset of B.

Proof: B can be covered by trivializing open subsets.
Hence $[0,1]$ can be covered by open subsets U such that $\gamma(U)$ lies in a trivializing open subset of B. One can actually choose these U's to be open intervals. $[0,1]$ being compact it can be covered by finitely many such open intervals. Hence the result. □.

We now prove the existence of $\tilde{\gamma}$. Define $\tilde{\gamma}(0) = e_0$.

- $\gamma([t_0, t_1]) \in U_0$ trivializing open: $\varphi_0 : p^{-1}(U_0) \xrightarrow{\sim} U_0 \times S_0$.
  Let $s_0 \in S_0$ be such that $\varphi_0(e_0) \in U_0 \times \{s_0\}$, and define
  $\tilde{\gamma}(t) = \varphi_0^{-1}(\gamma(t), s_0)$ for $0 = t_0 \leq t \leq t_1$.
- assume we have constructed $\tilde{\gamma}(t)$ for $0 = t_0 \leq t \leq t_i$.
  $\gamma([t_i, t_{i+1}]) \in U_i$ trivializing open: $\varphi_i : p^{-1}(U_i) \xrightarrow{\sim} U_i \times S_i$.
  Let $s_i \in S_i$ be such that $\varphi_i(\tilde{\gamma}(t_i)) \in U_i \times \{s_i\}$, and define
  $\tilde{\gamma}(t) = \varphi_i^{-1}(\gamma(t), s_i)$ for $t_i \leq t \leq t_{i+1}$.
- by induction we get a construction of the lift $\tilde{\gamma} : [0,1] \to E$ starting at e_0.

Unicity follows from the fact that we had no choice in the construction of $\tilde{\gamma}$.

- first of all, $\tilde{\gamma}(0)$ must be e_0.
- second of all, being a lift we must have $\tilde{\gamma}(t) = \varphi_0^{-1}(\gamma(t), s_0(t))$ for $0 \leq t \leq t_1$, with $s_0 : [0, t_1] \to S_0$ a continuous map. But S_0 is discrete and $[0, t_0]$ is connected, hence s_0 must be constant. ... □

Lemma: Let $p : E \to B$ be a covering map and let $e_0 \in E$. Any continuous map $F : [0,1]^2 \to B$ such that $F(0,0) = b_0 = p(e_0)$ can be uniquely lifted to an $\tilde{F} : [0,1]^2 \to E$ such that $\tilde{F}(0,0) = e_0$. Moreover, if F is a path-homotopy then so is $\tilde{F}$.

Proof: • the proof of the first part (existence and uniqueness of $\tilde{F}$) is similar to the previous Lemma: one first find a subdivision of $[0,1]^2$ into closed rectangles that are sent into trivializing open subsets via F, and then one proceed by induction (use the lexicographic order on the rectangles).

- if F is a path-homotopy then $F(\{0\}\times[0,1])=\{b_0\}$.

Hence $\tilde{F}(\{0\}\times[0,1])\subset p^{-1}(b_0)$. Thus $\tilde{F}(\{0\}\times[0,1])=\{e_0\}$.

(continuous connected discrete) $\Rightarrow$

The proof that $\tilde{F}(1,t)$ is constant is identical. □.

<u>Theorem</u>: let $p:E\to B$ be a covering map, let $e_0\in E$ and let $\gamma_0,\gamma_1:[0,1]\to B$ two paths starting at $b_0:=p(e_0)$. Let $\tilde{\gamma}_0,\tilde{\gamma}_1:[0,1]\to E$ be their respective lifts starting at e_0. If $\gamma_0(1)=\gamma_1(1)$ and $\gamma_0\sim\gamma_1$ then $\tilde{\gamma}_0(1)=\tilde{\gamma}_1(1)$ and $\tilde{\gamma}_0\sim\tilde{\gamma}_1$.

<u>Proof</u>: let F be a path-homotopy between γ_0 and γ_1, and let $\tilde{F}$ be its lift such that $\tilde{F}(0,0)=e_0$. We know from the previous lemma that $\tilde{F}$ is a path-homotopy. Finally observe that $p\circ\tilde{F}(t,i)=F(t,i)=\gamma_i(t)$ for $i=0$ or 1. Hence by unicity of lifts $\tilde{F}(t,i)=\tilde{\gamma}_i(t)$. □.

Under the above assumption we thus define a map

$$\Phi_p:\pi_1(B,b_0)\longrightarrow p^{-1}(b_0)$$
$$\gamma\longmapsto\tilde{\gamma}(1)\text{, where }\tilde{\gamma}\text{ is a lift of }\gamma\text{ starting at }e_0.$$

It follows from the above theorem that Φ_p is well-defined.

<u>Theorem</u>: (1) $p_*:\pi_1(E,e_0)\to\pi_1(B,b_0)$ is injective.

(2) let $H:=p_*(\pi_1(E,e_0))$. Then Φ_p descends to an injective map $\pi_1(B,b_0)/H\longrightarrow p^{-1}(b_0)$.

Moreover, it is bijective if E is path-connected.

(3) $[\gamma]\in H\Leftrightarrow\tilde{\gamma}$ is a loop at e_0.

<u>Proof</u>:
- (1) let γ be a loop at e_0 in E, and assume that there exists a path-homotopy F between $p_*\gamma$ and the constant loop at b_0. Then there exists a lift $\tilde{F}$ of F, which is a path-homotopy between γ (lift of $p_*\gamma$ starting at e_0) and the constant loop at e_0 (which lifts the constant loop at b_0).
- (3) Let γ be a loop based at b_0 in B, and let $[\gamma]$ its class in $\pi_1(B,b_0)$.

$[\gamma]\in H\Leftrightarrow\exists$ loop η at e_0 in E such that $\gamma\sim p_*\eta$.

$\Leftrightarrow\exists$ loop η at e_0 in E such that $\tilde{\gamma}\sim\eta$ (where $\tilde{\gamma}$ is the unique lift of γ starting at e_0)

$\Leftrightarrow\tilde{\gamma}$ is a loop at e_0 in E.

• (2) We start with a very useful fact:

<u>Claim</u>: let $\gamma : [0,1] \to B$ with $\gamma(0) = b_0$ and $\gamma(1) = b_1$.

let $\eta : [0,1] \to B$ with $\eta(0) = b_1$.

let $\tilde{\gamma}$ the lift of γ starting at e_0.

let $\tilde{\eta}$ the lift of η starting at $e_1 := \tilde{\gamma}(1)$.

Then $\tilde{\gamma}^{-1}$ is the lift of γ^{-1} starting at e_1 and $\tilde{\gamma} * \tilde{\eta}$ is the lift of $\gamma * \eta$ starting at e_0. We write $\tilde{\gamma}^{-1} = \widetilde{\gamma^{-1}}$ and $\tilde{\gamma} * \tilde{\eta} = \widetilde{\gamma * \eta}$.

<u>Proof</u>: immediate. It follows from the unicity of lifts. □

→ we now prove that Φ_p is well-defined on $\pi_1(B, b_0)/_H$.

If $[\gamma] \in H$ and $\eta : [0,1] \to B$ is a loop at b_0, then

$$\Phi_p([\gamma] * [\eta]) = \Phi_p([\gamma * \eta]) = \tilde{\gamma} * \tilde{\eta}(1) = \tilde{\eta}(1) = \Phi_p([\eta]).$$

because: 1) $[\gamma] \in H \Rightarrow \tilde{\gamma}$ ends at e_0 (hence $\tilde{\gamma} * \tilde{\eta}$ makes sense)

2) $\widetilde{\gamma * \eta} = \tilde{\gamma} * \tilde{\eta}$.

→ we turn to the proof of injectivity.

if $\Phi_p([\gamma]) = \Phi_p([\eta])$ then $\tilde{\gamma}$ and $\tilde{\eta}$ both end at the same point

Hence $\tilde{h} := \widetilde{\gamma * \eta^{-1}} = \tilde{\gamma} * \tilde{\eta}^{-1}$ is a loop based at e_0.

Therefore $[\tilde{\gamma}] = [\tilde{h}] * [\tilde{\eta}]$ and thus $[\gamma] = [p_* \tilde{h}] * [\eta] \in H * [\eta]$.

→ finally, if E is path-connected then for any $e \in p^{-1}(b_0)$ there exists a path γ from e_0 to e. Hence $\Phi_p([p_* \gamma]) = e$.

Φ_p is thus surjective. □