Topology

D-MATH, FS 2013 Damien Calaque

Exercice sheet 5

Some properties of metric spaces

1. Show that every point in a metric space S has a countable basis of neighbourhoods[1] ;
2. Show that on a metric space (S, d), the induced topology is the coarsest such that the distance is a continuous function $d : \mathrm{S} \times \mathrm{S} \to \mathbb{R}$.

Ultrametrics : a metric space (S, d) is said to be *ultrametric* if and only if for every $x, y, z \in \mathrm{S}$,

$$d(x, z) \leq \mathrm{Max}(d(x, y), d(y, z))$$

1. Let S be a set and d be the trivial distance on S. Show that (S, d) is ultrametric ;
2. Show that in an ultrametric space, every triangle is isosceles and that every point of an open ball is a center of this open ball ;
3. Let p be a prime number and for a natural number n, define

$$v_p(n) = \mathrm{Sup}\{k \in \mathbb{N} \,|\, p^k \text{ divides } n\}.$$

Then for a rational number $r = \frac{t}{q}$ define $v_p(r) = v_p(t) - v_p(q)$. For any two rationals x, y, set $d_p(x, y) = e^{-v_p(x-y)}$.

Show that $(\mathbb{Q}, d_p)$ is ultrametric.

Metrizable or not metrizable ?

Using the very first exercise or using the fact that in a metric space the sequential continuity implies continuity, answer the following questions.

1. Is the box topology on $\mathbb{R}^{\mathbb{N}}$ metrizable ?
2. Let I be a non countable set, is the product topology on $\mathbb{R}^{\mathrm{I}}$ metrizable ?

ℓ_p spaces

1. Let $(\mathrm{S}_1, d_1), ..., (\mathrm{S}_n, d_n)$ be a finite number of metric spaces. Show that for every $p \geq 1$, the function

$$D_p(x, y) = \left(\sum_{1 \leq i \leq n} d_i(x_i, y_i)^p\right)^{\frac{1}{p}}$$

is a metric which induces the product topology on $\mathrm{S}_1 \times ... \times \mathrm{S}_n$;

[1] A basis of neighbourhoods of x is a subset $\mathcal{B}$ of $\mathcal{V}_x$, the set of all neighborhoods of x, such that any element of $\mathcal{V}_x$ contains an element of $\mathcal{B}$.

2. Let S be the subset of $\mathbb{R}^{\mathbb{N}}$ consisting of the sequences with finitely many non-zero terms. For $p \geq 1$ and any two sequences $(\underline{x}, \underline{y}) \in \mathrm{S}$ define

$$d_p(\underline{x}, \underline{y}) = \left(\sum_{n=0}^{\infty} (x_n - y_n)^p \right)^{\frac{1}{p}}$$

Show that (S, d_p) is a metric space and compare the topologies induced on S by the different d_p.

Due on Thursday, March 28, 2013