Topology

D-MATH, FS 2013

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Exercise Sheet 8

1. *(Topological groups)*

(a) Prove that the map $m : O_n(\mathbb{R}) \times O_n(\mathbb{R}) \to O_n(\mathbb{R})$ sending the pair (A, B) to AB is a continuous map.

(b) Prove that the map $i : O_n(\mathbb{R}) \to O_n(\mathbb{R})$ sending A to A^{-1} is a continuous map.

Remark
A group which is a topological space such that the maps m (the product) and i (the map sending every element to its inverse) are continuous is called *topological group*.

2. (a) Prove that a topological space X is Hausdorff if and only if the diagonal $\Delta := \{(x, x) \mid x \in X\} \subset X \times X$ is closed.

(b) Show that a topological group (see remark above) X is Hausdorff if and only if $\{e\}$ is closed.

3. *(Torus)* Let T_1 be the topological space defined by the quotient of $\mathbb{R}^2$ modulo the following relation

$$(x, y) \sim (x', y') \iff \exists\, (p, q) \in \mathbb{Z}^2 \text{ such that } x' = x + 2\pi p, y' = y + 2\pi q$$

(a) Let $R, r \in \mathbb{R}$ such that $R > 2r$. Consider the map $p : [0, 2\pi) \times [0, 2\pi) \to \mathbb{R}^3$ defined by:

$$p(\theta, \phi) := \big((R + r\cos(\phi))\cos(\theta), (R + r\cos(\phi))\sin(\theta), r\sin(\phi)\big).$$

Let T_2 be the topological space $p\big([0, 2\pi) \times [0, 2\pi)\big) \subset \mathbb{R}^3$. Show that

(i) The map p is not a homeomorphism onto its image.

(ii) Show that T_2, T_1, and $S^1 \times S^1$ are homeomorphic.

(b) Show that T_2, T_1 are connected and compact.

(c) Show that T_1 has the structure of a topological group where

$$m\big((x_1, y_1)(x_2, y_2)\big) := (x_1 + x_2, y_1 + y_2) \quad i(x, y) := (-x, -y),$$

and where the unit e is equal to the equivalence class of the point $(0, 0)$.

4. *(Sphere)* Let $S^2 := \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3$.

(a) Show that S^2 is homeomorphic to the Alexandroff compactification (see ex. sheet 7) of $\mathbb{R}^2$.

(b) Show that S^2 is not homeomorphic to $S^1 \times S^1$. One can use the following fact

Jordan curve theorem on S^2.
Let $\gamma : S^1 \to S^2$ be a simple closed curve, i.e. $\gamma : S^1 \to \gamma(S^1)$ is a homeomorphism. Then $S^2 \backslash \gamma(S^1)$ has two connected components.

5. *Analysis results*

(a) Show that on any great circle around the world, for the temperature, pressure, elevation, carbon dioxide concentration (or any other similar scalar quantity which varies continuously) there always exist two antipodal points that share the same value for that variable.

(b) *Dini's Theorem.* Let X be a compact metric space and consider a monotonically increasing sequence $f_n : X \to \mathbb{R}$ of continuous functions on X. I.e. we assume $f_n(x) \leqslant f_{n+1}(x)$ for all $n \in \mathbb{N}$ and $x \in X$. Show that if the sequence f_n converges pointwise to a continuous function $f : X \to \mathbb{R}$ then the convergence is uniform.

Due on Thursday, April 25, 2013