Topology
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Solution sheet 4

Some practical exercices

1. Because of the density of $\mathbb{Q}$ in $\mathbb{R}$, the only $\mathbb{Q}$-invariant open sets in $\mathbb{R}$ are the empty set and $\mathbb{R}$, so the quotient topology on $\mathbb{R}/\mathbb{Q}$ is the trivial topology.

2. $\underline{\alpha \in \mathbb{Q}}$, write $\alpha = \frac{p}{q}$ with $p \wedge q = 1$, then $\mathbb{Z} + \alpha\mathbb{Z} = \frac{1}{q}(q\mathbb{Z} + p\mathbb{Z}) = \frac{1}{q}\mathbb{Z}$. Set $\varphi : x \mapsto \frac{q}{2\pi}x$, then φ is a homeomorphism of $\mathbb{R}$ that induces a homeomorphism between the descrete subspaces $2\pi\mathbb{Z}$ and $q\mathbb{Z}$. This means that φ induces a homeomorphism between $\mathbb{R}/(2\pi\mathbb{Z})$ and $\mathbb{R}/(\mathbb{Z} + \alpha\mathbb{Z})$. Now $\mathbb{R}/(2\pi\mathbb{Z})$ is homeomorphic to the circle ; the quotient topology on $\mathbb{R}/(\mathbb{Z}+\alpha\mathbb{Z})$ is then Hausdorff.

3. Let P_1 and P_2 be two non zero polynomial in $\mathbb{C}[\mathrm{X}, \mathrm{Y}]$.
Because P_1 and P_2 are two non zero polynomial of finite degree, it is possible to find $a \in \mathbb{C}$ such that $\mathrm{P}_1(a, \mathrm{Y})$ and $\mathrm{P}_2(a, \mathrm{Y})$ are non zero polynomial in one variable. But, any polynomial of one variable has a finite number of roots and $\mathbb{C}$ is infinite, so there exists $b \in \mathbb{C}$ such that $\mathrm{P}_1(a, b) \neq 0$ and $\mathrm{P}_2(a, b) \neq 0$.
We deduce that any two non empty open sets have a non trivial intersection ; the topology is not Hausdorff.

4. Let u and v be two non zero vectors in $\mathbb{R}^3$. Note θ the plane angle between u and v. Note C_u the open filled cone with axis u and angle $\theta/2$ and C_v the one with axis v. Then these are two disjoint open subset of $\mathbb{R}^3\backslash 0$ with disjoint open images in the quotient. And they separate the images of u and v. Hence the quotient topology is Hausdorff.

Universal property of the quotient topology

Because of a general property of the quotient in Set Theory, we know it exists a unique map $\varphi : \mathrm{S}/\mathrm{R} \to \mathrm{T}$ such that $f = \varphi \circ q$.
Let's show φ is continuous. For this, let V be an open set in T. Then $\mathrm{U} = f^{-1}(\mathrm{V}) = q^{-1} \circ \varphi^{-1}(\mathrm{V})$. Because f is supposed continuous U is an open set in S. But the open sets O in the quotient are precisely those for which $q^{-1}(\mathrm{O})$ is an open set of S. We deduce that $\varphi^{-1}(\mathrm{V})$ is an open set of the quotient, hence φ is continuous.

When is the quotient topology Hausdorff ?

1. $\Rightarrow$: suppose the quotient topology is Hausdorff. Take x and y in S/R two distinct points and U, V two disjoint open sets containing respectiveley

x and y. Then $q^{-1}(\mathrm{U})$ and $q^{-1}(\mathrm{V})$ are disjoint saturated open sets in S containing respectively the equivalent class of x and y.
$\Leftarrow$: let x and y be two distinct points in S/R. And let U and V be two disjoint saturated open subset of S containing respectively the equivalent class of x and y. Then, $q(\mathrm{U})$ and $q(\mathrm{V})$ are disjoint open sets in S/R separating x and y. So the quotient topology is Hausdorff.

2. Let (x, y) be a point in $(\mathrm{S} \times \mathrm{S}) \backslash \mathrm{graph}(\mathrm{R})$. Then the image of x and y are distinct in S/R, so by 1. I can find two disjoint open sets U and V of S with disjoint images in S/R such that U and V contains respectively the equivalent class of x and y. Thus, $\mathrm{U} \times \mathrm{V}$ is an open neighborhood of (x, y) in $\mathrm{S} \times \mathrm{S}$ with empty intersection with graph(R) ; the graph is closed.

3. Let x and y be two points in S such that $q(x) \neq q(y)$. Then (x, y) in not in the graph of R. As the graph of R is closed, it is possible to find an open neighborhood of (x, y) in $\mathrm{S} \times \mathrm{S}$ that do not intersect the graph of R. Furthemore, because $\{\mathrm{U} \times \mathrm{V} | \mathrm{U}, \mathrm{V} \subset \mathrm{Open}(\mathrm{S})\}$ is a basis of the product topology on $\mathrm{S} \times \mathrm{S}$, we can suppose that this open set is $\mathrm{U} \times \mathrm{V}$ for U, V two open subsets of S.
Now because the map q is open, $q(\mathrm{U})$ and $q(\mathrm{V})$ are disjoint open subsets of S/R separating $q(x)$ and $q(y)$. So the quotient topology is Hausdorff.

Open and closed maps

1. Consider $f : \mathbb{R} \to \{x\}$ the projection to a point and $f' = \mathrm{Id}_{\mathbb{R}}$; f and f' are closed maps. But $f \times f'$ is not closed. For example, the hyperbola $\mathcal{H} = \{ab = 1 | a, b \in \mathbb{R}\}$ in $\mathbb{R}^2$ is closed, but $(f \times f')(\mathcal{H}) = \{x\} \times \mathbb{R}^*$ which is not closed in $\{x\} \times \mathbb{R}$ for the product topology.

2. We can check that $q : \mathrm{A} \to \mathbb{R}$ is a surjective map and for any open subset U of $\mathbb{R}$, $q^{-1}(\mathrm{U}) = \pi^{-1}(\mathrm{U}) \cap \mathrm{A}$, it is an open subset of A by definition of the induced topology on A. Thus q is a quotient map.

To see that this map is neither open nor closed, consider the subsets of A : $\mathrm{B} = \mathrm{A} \backslash (\{(x, y) \in \mathrm{A} | x \leq 0\} \cup \{(0, 0)\})$ and $\mathrm{C} = \mathrm{A} \backslash \{(x, y) \in \mathrm{A} | x^2 + y^2 < 1\}$.