Solution sheet 5

A bit of Cardinal Theory : a *cardinal* is an equivalence class of sets under the equivalence relation $X \sim Y$ if and only if there exists a bijection between X and Y. The cardinal of X will be noted $|X|$ and we say that $|X| \leq |Y|$ if and only if there exists an injection from X to Y.
Examples : $|\mathbb{N}| = |\mathbb{Q}|$ since there exists a bijection from $\mathbb{N}$ to $\mathbb{Q}$. But $|\mathbb{Q}| < |\mathbb{R}|$ since there exists an injection of $\mathbb{Q}$ in $\mathbb{R}$ but no bijection between them.
Notations : the first non-finite cardinal, is the cardinal of $\mathbb{N}$ and is called $\aleph_0$; the cardinal of the set of functions from X to Y will be noted $|Y|^{|X|}$, so in particular, $\mathfrak{P}(X)$ is noted $2^{|X|}$. Finally, the cardinality of $X \times Y$ is noted $|X||Y|$.
Main theorem of Cardinal Theory : for every set X, $|X| < 2^{|X|}$.

Some properties of metric spaces

1. Let x be a point of S and V be a neighbourhood of x. Then by definition of the metric topology, there exists $\varepsilon > 0$ such that $B(x, \varepsilon) \subset V$. Choose $n \in \mathbb{N}^*$ such that $\varepsilon > \frac{1}{n}$, then $B(x, \frac{1}{n}) \subset V$. As a consequence, $\{B(x, \frac{1}{n}), n \in \mathbb{N}^*\}$ is a countable basis of neighbourhoods of x.
2. Let $\mathcal{T}$ be a topology on S such that d is continuous. In particular, for every point $x \in S$, $d(x, .) : S \to \mathbb{R}$ is continuous and so all the balls $B(x, r) = d(x, .)^{-1}(]-r, r[)$ with $r > 0$ are in $\mathcal{T}$.

But because the open balls form a basis of the metric topology on S, the topology induced by the metric is the coarsest on S such that d is continuous.

Ultrametrics.

1. Let x, y, z be in S, if $x = z$, then $d(x, z) = 0$ and there is nothing to prove and if $x \neq z$, then y cannot be equal to both x and z, so say it is different from z, then $d(x, z) = 1$ and $\mathrm{Max}(d(x, y), d(y, z)) = 1$ and the inequality is satisfied.
2. Let x, y, z be points in an ultrametric space (S, d), then we can suppose $d(x, y) \geq d(y, z) \geq d(z, x)$. Then by the ultrametric inequality, we have $d(x, y) \leq \mathrm{Max}(d(y, z), d(z, x)) = d(y, z)$, so $d(x, y) = d(y, z)$ and the triangle xyz is isoceles.

Let $x \in S$ $r > 0$, and $y \in B(x, r)$. Then for every $z \in B(x, r)$, we have $d(z, y) \leq \mathrm{Max}(d(y, x), d(z, x)) < r$. So y is a center of the ball $B(x, r)$.
3. Remark that because $v_p(x - y) \in [0, +\infty]$, $d_p(x, y) \in [0, 1]$ for $x, y \in \mathbb{Q}$. Moreover d_p is symmetric because v_p is symmetric. If $x, y \in \mathbb{Q}$ are such that $d_p(x, y) = 0$, then $v_p(x - y) = \infty$ which means that $x = y$. Finally,

let $x, y, z \in \mathbb{Q}$, because $x - y = (x - z) + (z - y)$, if p^n divides both $x - z$ and $z - y$, then p^n divides $x - y$. So $v_p(x - y) \geq \mathrm{Min}(v_p(x - z), v_p(z - y))$. This implies that $d_p(x, y) \leq \mathrm{Max}(d_p(x, z), d_p(z, y))$. Hence $(\mathbb{Q}, d_p)$ is an ultrametric space.

Metrizable or not metrizable ?

1. Let x be a point in $\mathbb{R}^{\mathbb{N}} = \mathbb{R}_0 \times \mathbb{R}_1 \times ... \times \mathbb{R}_n \times ....$ Then any basis of neighbourhood of x for the box topology has the form $\mathcal{B}_0 \times \mathcal{B}_1 \times ... \times \mathcal{B}_n \times ...$ where $\mathcal{B}_i$ is a basis of neighbourhoods of x_i.

Now for the standard topology on $\mathbb{R}$, every basis of neighbourhoods of x_i is infinite. Indeed, suppose it is finite, let V be the intersection of all the elements of the basis ; because the intersection is finite, it is still a neighbourhood of x_i. Then take an open ball $\mathrm{B}(x, r) \subset \mathrm{V}$ for r sufficiently small so that it is strictly contained in V. This open ball is not containing any element of the basis.

Conclusion, any basis of neighbourhood of a point in $\mathbb{R}^{\mathbb{N}}$ for the box topology has at least the cardinality $\aleph_0^{\aleph_0} > 2^{\aleph_0} > \aleph_0$, so by the first exercise the box topology is not metrizable.

2. Let $x \in \mathbb{R}^{\mathrm{I}}$. For every $i \in \mathrm{I}$, a basis of neighbourhoods of x_i has at least cardinality $\aleph_0$, as said in 1. Now a basis of neighbourhoods of x for the product topology is made of sets of the form $\mathbb{R} \times ... \times \mathrm{V}_{i_1} \times ... \times \mathbb{R} \times ... \times \mathrm{V}_{i_n} \times \mathbb{R} \times \mathbb{R}...$ for a finite n, with $\mathrm{V}_{i_1}, ...\mathrm{V}_{i_n}$ elements of the basis of $x_{i_1}, ..., x_{i_n}$. So any basis of neighbourhoods of x has cardinality at least $\aleph_0 \times \mathrm{I} > \aleph_0$, so again by the first exercise the product topology on $\mathbb{R}^{\mathrm{I}}$ is not metrizable.

ℓ_p spaces

1. <u>D_p is a metric</u> : let x, y be in the product, then $\mathrm{D}_p(x, y) \in [0, +\infty)$ because it is a finite sum of finite non negative real numbers. Then $\mathrm{D}_p(x, y) = 0$ if and only if the sum of all the $d_i(x, y)$ is zero, meaning that each $d_i(x, y)$ has to be zero and so for every i, $x_i = y_i$ i.e $x = y$. And because the d_i are symmetric, D_p is symmetric.

Let z be another element of the product. Then for every i, $d_i(x, y) \leq d_i(x, z) + d_i(z, y)$, so by direct calculation and because $x \mapsto x^p$ and $x \mapsto x^{1/p}$ are non decreasing functions, we have $\mathrm{D}_p(x, y) \leq \mathrm{D}_p(x, z) + \mathrm{D}_p(z, y)$.

So D_p is a metric on the product.

<u>All the topologies induced by the D_p are the same</u> : let x be a point in the product and $r > 0$, then

$$\prod_{1 \leq i \leq n} \mathrm{B}_{d_i}\left(x_i, \frac{r}{n^{1/p}}\right) \subset \mathrm{B}_{\mathrm{D}_p}(x, r).$$

So the product topology is finer that the topology induced by D_p. Conversely,

$$\mathrm{B}_{\mathrm{D}_p}(x,r) \subset \prod_{1\leq i\leq n} \mathrm{B}_{d_i}(x_i,r)$$

so the induced topology is finer than the product topology. Hence, they are the same.

2. d_p is a metric : let x,y,z be elements of S. By definition, there exists N such that for any $n \geq \mathrm{N}$, $x_n = y_n = z_n$. So all the statements of x,y,z and d_p that we want to make are the same than the previous ones with D_p and $\mathbb{R}^{\mathbb{N}}$. In particular, d_p is a metric on S.

Comparision : let $q > p$, $x \in \mathrm{S}$ and $r < 1$, then

$$\mathrm{B}_p(x,r) \subset \mathrm{B}_q(x,r^{p/q})$$

So the topology induced by d_p is finer than the one induced by d_q.

The converse if false, for every $r, \varepsilon < 1$, choose $\mathrm{N} \in \mathbb{N}$ and $a \in \mathbb{R}$ such that $\mathrm{N}a^q < r^q$ and $\mathrm{N}a^p > \varepsilon^p$. Then the sequence with the N first terms equal to a is in $\mathrm{B}_q(0,r)$ but not in $\mathrm{B}_p(0,\varepsilon)$.

Conclusion : the topology induced by d_p is strictly finer than the one induced by d_q.