Solution sheet 6

Some examples

1. Let S be a topological space and R be an equivalence relation on S. Then let f be a continuous function from S/R to $\{0,1\}$. This induces a continuous function $\tilde{f} : S \to \{0,1\}$, which has to be constant by connectedness of S. So f is constant too and S/R is connected.
2. Let $\phi : \mathbb{R} \to \mathbb{R}^2$ be a homeomorphism. Then by ϕ, $\mathbb{R}^*$ is homeomorphic to $\mathbb{R}^2 - \{\phi(0)\}$. But $\mathbb{R}^*$ is not path-connected whereas $\mathbb{R}^2 - \{\phi(0)\}$ is path connected, contradiction.

Product of connected spaces.

1. The topological space S_K is homeomorphic to $\prod_{i \in K} S_i$ which is a finite product of connected spaces, hence connected.
2. The intersection $S_K \cap S_{K'}$ is never empty and all the S_K are connected, so Y is connected.
3. Let y be any element of S and U an open neighbourhood of y. Then U contains a element of the basis of the product topology i.e $\prod_{i \in I} U_i$ with K a finite subset of I and $U_i = S_i$ for all $i \notin K$. The intersection $S_K \cap \prod_i U_i$ is not empty. So the closure of Y is equal to S. The closure of a connected space is connected, so S is connected.

Connected or not connected ?

The spaces 1, 4, 5 are locally path-connected. For them, being connected is equivalent to being path-connected.

1. $\mathbb{R}^*$ is not connected because $(-\infty, 0)$ and $(0, +\infty)$ are two non-empty disjoint open sets covering $\mathbb{R}^*$. Now those two open sets are connected because they are convex, hence they are the two connected components.
2. We will show that the connected component of any rationnal x, is $\{x\}$ and so, $\mathbb{Q}$ is not connected (in fact, it is *totally disconnected*). Suppose that C_x is the connected component of x. Then if C_x has at least two points $a < b$, choose any irrational $a < r < b$; then $(-\infty, r) \cap C_x$ and $(r, +\infty) \cap C_x$ are open non-empty and separates C_x. So the connected components are the points.
3. The topological space $S = \{(x, \sin(\frac{1}{x})) | x > 0\} \cup \{(0, y) | y \in [-1, 1]\}$ is the closure of the graph of the function $x \mapsto \sin(\frac{1}{x})$ on $(0, 1]$. The graph of a continuous function on a connected space is connected ; so S is the closure of a connected space, hence it is connected. But it is not path-connected. For example, if it were possible to connect $(0, 0)$ with $(1, \sin 1)$ by a path, it would coincide on $(0, 1]$ with the function $x \mapsto \sin(\frac{1}{x})$ and so, it would mean that this function has a limit when $x \to 0$, which is not the case.

4. The determinant function is a continuous function on $O_n(\mathbb{R})$ with values in $\{-1, 1\}$ which is not constant. Hence $O_n(\mathbb{R})$ is not connected. The two connected components of $O_n(\mathbb{R})$ are SO_n and O_n^-. Indeed, by the decomposition theorem of orthogonal matrices in blocks of the form $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$, SO_n is homeomorphic to $(SO_2)^{\lfloor \frac{n}{2} \rfloor}$. The space SO_n is known to be homeomorphic to the circle which is connected, hence SO_n is connected because it is homeomorphic to a product of connected spaces. Finally, by multiplication by $\mathrm{diag}(-1, 1, 1, ..., 1)$ we see that O_n^- is homeomorphic to SO_n and thus, is connected.
5. The morphism $\phi : (a, b) \mapsto \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$ is a bijection from S^3 to SU_2 whose inverse is $\phi^{-1} : U \mapsto (\mathrm{pr}_{11}(U), \mathrm{pr}_{12}(U))$. Both are continuous so it is a homeomorphism. Hence SL_2 is connected.

Application to Analysis

1. The set A is defined by an equality, so it is closed. Now let x be an element of A, by the mean property, for $r > 0$ sufficiently small,

$$f(x) = \frac{1}{2\pi} \int_0^{2\pi} f(x + re^{i\theta}) d\theta \leq \sup_\Omega f = f(x)$$

this implies that $f(y) = f(x)$ for y sufficiently near x, so A is open.
2. As A is non empty by assumption, open and closed in Ω connected, we have $A = \Omega$ and f is constant.

A riddle

Let f be such a function, then because $\mathbb{Q}$ is countable, $f(\mathbb{R})$ is a countable subset of $\mathbb{R}$. Because $\mathbb{R}$ is connected, $f(\mathbb{R})$ is a connected countable subset of $\mathbb{R}$, hence a point. Conclusion : such a function cannot exist.

Pathology : the ordered square

1. Suppose $c \in B$, then $c \neq a$, so either $c = b$ or $a < c < b$. In either cases, it follows that an interval $(d, b]$ is contained in B. If $c = b$ we have a contradiction at once, for d is a smaller upper bound on A than c. If $c < b$, we note that $(c, d]$ does not intersect A (because c is an upper bound on A). Then $(d, b] = (d, c] \cup (c, b]$ does not intersect A. Again d is a smaller upper bound on A than c, contrary to construction. So $c \notin B$.

Suppose now $c \in A$, then $c \neq b$, so either $c = a$ or $a < c < b$. Because A is open in $[a, b]$, there must be some interval of the form $[c, e)$ contained in A. Because of order topology of the linear continuum L, we can choose a point z of L such that $c < z < e$. Then $z \in A$, contrary to the fact that c is an upper bound for A. So $c \notin A$.

conclusion : $[a, b]$ is connected.

3. Connected sets are convex in the order topology. Now suppose that C is convex and that A and B are two non empty disjoint open sets covering C. Choose $a \in$ A and $b \in$ B, then $[a, b] \cap$ A and $[a, b] \cap$ B are disjoint open sets covering $[a, b]$ which is connected, so one of the two has to be empty, for example A. This is a contradiction because $a \in$ A. So C is connected. Hence the connected subsets of L are exactly the convex ones.

4. The square S is a linear continuum. Indeed, for the order topology, for every $x < y$ in S it is possible to find $x < z < y$. Let's check it has the least-upper-bound property. Let π be the projection map to the first factor. It is continuous (w.r.t the product topology on S) and surjective. Let A be a subset of S which is bounded above. Consider $\pi(\mathrm{A})$. Since A is bounded above, $\pi(\mathrm{A})$ is bounded above and as it is a subset of I it has the least-upper-bound property. Let b be the least upper bound, if b belongs to $\pi(\mathrm{A})$, then $b \times$ I will intersect A at say $b \times c$ for some $c \in$ I. Notice that since $b \times$ I has the same order type of I, the set $(b \times I) \cap$ A will indeed have a least upper bound $b \times c'$, which is the least upper bound for A.

If b doesn't belong to $\pi(\mathrm{A})$, then $b \times 0$ is the least upper bound of A, for if $d < b$, and $d \times e$ is an upper bound of A, then d would be a smaller upper bound of $\pi(\mathrm{A})$ than b, contradicting the unique property of b.

5. Suppose the ordered square is locally path-connected, then because it is connected, it will be path-connected. So let $f : [0, 1] \to$ S be a path from $(0, 0)$ to $(1, 1)$. The image set $f([a, b])$ must contain every point of S by the intermediate value theorem. Therefore, for each $x \in$ I, the set $\mathrm{U}_x = f^{-1}(x \times (0, 1))$ is a non empty subset of $[0, 1]$; by continuity, it is open in $[0, 1]$. Then $[0, 1]$ contains an uncountable disjoint union of non-empty open sets, which is not possible.

In short : the space $[0, 1]$ has a countable basis of neighbourhoods, which is not the case of S.

Due on Thursday, Mai 4, 2013