Topology

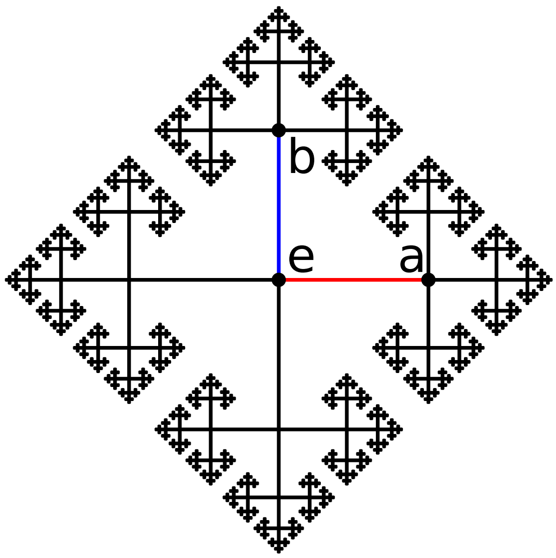
D-MATH, FS 2013

Damien Calaque

Solutions of exercise sheet 12

Using Topology for Algebra: The goal of this sheet is to show that subgroups of free groups are themselves free using algebraic topology.

1. (a) Let x be the point where the two circles are glued together. Fix loops a and b at x, going once around the first and second circle, respectively. For any preimage e of x under a covering map, it must be possible to lift a, b, a^{-1} and b^{-1} to paths in the covering space, and if the space is to be simply connected, these paths cannot be loops, but must lead to different preimages of x. These observations motivate the following covering space E:



The covering map $p : E \to B_2$ maps the vertices to x, and the edges are mapped to the respective loops at x. Note that the topology on E is the quotient topology identifying each edge with $[0, 1]$ and gluing them together.

More explicitely, E is the *Cayley graph* of F_2: The vertices of E are the elements of F_2, and, at each vertex $g \in F_2$, insert two egdes joing g to ga and gb, respectively. Note that F_2 acts on the Cayley graph: a shifts to the right along the central horizontal axis and B shifts upward along the central vertical axis. This graph is simply connected because any loop would give us a relation between the generators a and b.

(b) Using the properties of the Cayley graph mentioned above, the group of deck transformations $\mathcal{D}(p)$ of the $p : E \to B_2$ is F_2, the free group on two elements. Thus,

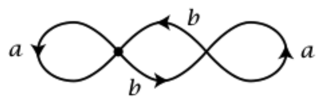
$$\pi_1(B_2) \cong \mathcal{D}(p) \cong F_2,$$

(c) The wedge sum $\bigvee_\alpha X_\alpha$ of a collection of spaces X_α with basepoints $x_\alpha \in X_\alpha$ is the quotient space of the disjoint union $\bigsqcup_\alpha X_\alpha$ in which all basepoints x_α are identified to a single point. If we take $X_\alpha = S^1$ for every α, every basepoint x_α is a deformation retract of an open neighborhood U_α in X_α, and X_α is a deformation retract of its open neighborhood $A_\alpha = X_\alpha \bigvee_{\beta\neq\alpha} U_\beta$. Moreover, the intersection of two or more of the A_α's is $\bigvee_\alpha U_\alpha$, which deformation retracts to a point. Van Kampen's Theorem then implies that $\Phi : \ast_\alpha \pi_1(X_\alpha) \to \pi_1(\bigvee_\alpha X_\alpha)$ is an isomorphism, and, as $\pi_1(S^1)$ is a free group on one element, $\pi_1(\bigvee_\alpha S^1)$ is a free group, the free product of copies of $\mathbb{Z}$, one for each circle. In particular, $\pi_1(B_n) = F_n$.

2. (a) H is compact as it is a closed and bounded subspace of $\mathbb{R}^2$, wheras the wedge sum B_∞ is not compact: the complement of the distinguished point is a union of open intervals; to those add a small open neighborhood of the distinguished point to get an open cover with no finite subcover.

(b) One can show that the fundamental group of H is uncountable: the idea is that in H, there is a path running through all the (infinitely) many circles in order of decreasing radius (parametrizing the first circle on $[0, \frac{1}{2}]$, the second on $[\frac{1}{2}, \frac{3}{4}]$ and so on), and to any $\{0, 1\}$-sequence (a_n), we can produce a path in H by "skipping" the n-th circle (i.e. just remaining at the origin), whenever $a_n = 0$. It is not difficult to show that paths induced in this fashion are not homotopic if they were induced by different sequences. Therefore, we have found an inclusion $2^{\mathbb{N}} \to \pi_1(H, *)$. On the other hand, $F_\infty = \ast_{n\in\mathbb{N}}\mathbb{Z}$ is countable: indeed, every element of F_∞ is a finite word in the alphabet $\{1_n, (-1)_n : n \in \mathbb{N}\}$, where 1_n is a generator of the n-th copy of $\mathbb{Z}$ in the free product; and some standard juggling with bijections shows that the set of such words is countable (for instance, one can construct an injection into $\bigcup_{n=1}^\infty \mathbb{N}^n$ or into the set of finite subsets of $\mathbb{N}$). From this we can deduce that H is not homeomorphic to countably many copies of S^1 glued together at one point, since the fundamental group of that space is F_∞.

3. There is a covering $p : E \to B_2$ defined by



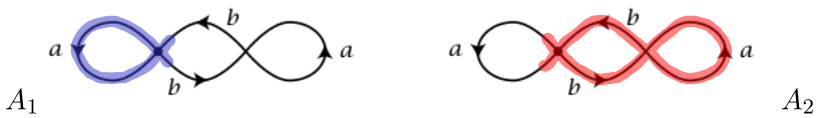
By the homotopy lifting lemma this induces an injection

$$p_* : \pi_1(E) \to \pi_1(B_2),$$

so $\pi_1(E)$ is isomorphic to a subgroup of $\pi_1(B_2) = F_2$.

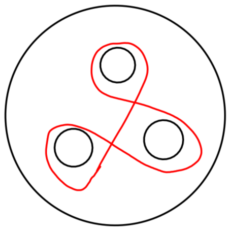
We can compute the fundamental group of E in two different ways:

(a) We use Van Kampen's theorem as we did in exercise 1 (c) by using the cover $A_1 \cup A_2$, where A_1 covers a little bit ("contractibly") more than the left loop and A_2 covers a little bit more ("contractibly") than the middle part and the right loop.

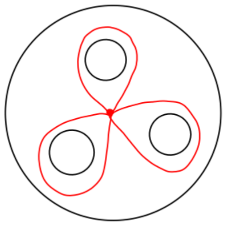


The intersection deformation retracts to a point, and $\pi_1(A_2) = F_2$, so $\pi_1(E) = F_3$.

(b) Another way is to see that E is a deformation retract of a disk with 3 points removed:



and B_3 also is a deformation retract of a disk with 3 points removed:



So $\pi_1(E) \cong \pi_1(B_3) \cong F_3$.

4.

Claim 1. *Every connected graph X contains a maximal subtree Y.*

The proof can be found in Hatcher, Algebraic Topology, Proposition 1A.1.

Claim 2. *Γ is homotopy equivalent to Γ/T, where T is the maximal subtree from the previous claim.*

The proof of this claim and the conclusion are Proposition 1A.2 in Hatcher.

5. E is graph with vertices $\bigcup_{v \text{ vertex of } \Gamma} p^{-1}(v)$ and edges the lifts of the edges of Γ: Let Γ^0 be the set of vertices of Γ. Then the set of vertices of E is $E^0 = p^{-1}(\Gamma^0)$. We can write Γ as a quotient space of $\Gamma^0 \bigsqcup_\alpha I_\alpha$, where $I_\alpha = [0,1]$. Applying the path lifting property to the maps $I_\alpha \to \Gamma$, we get unique lifts $I_\alpha \to E$, which define the edges of the graph E. As $p : E \to \Gamma$ is a local homeomorphism, the topology on E is the same as the one just defined as a graph.

It remains to show that E is locally finite if Γ is locally finite. Consider a vertex $v \in E^0$. Since Γ is locally finite, there are only finitely many edges at $p(v)$. Moreover, there is a neighborhood U of v such that $p|_U : U \to p(U) \subset \Gamma$ is a homeomorphism, so there are also only finitely many edges at v.

6. Let $F = \ast_{\alpha \in I} \mathbb{Z}$. Choose a connected graph Γ with $\pi_1(\Gamma) = F$, e.g. $\bigvee_{\alpha \in I} S^1$. Using the theorem, for any subgroup $G < F$ there is a covering space $p : E \to \Gamma$ with $p_*(\pi_1(E)) = G$. Since coverings induce injections on the corresponding fundamental groups, $\pi_1(E) \cong G$. By exercise 5, E is a graph and by exercise 4 G is free (it is connected since Γ is connected).

Due on Thursday, May 30, 2013