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Numerical Methods for Partial Differential Equations

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D-MATH

Homework Problem Sheet 3

Introduction. This assignment is fully theoretical and involves some training with inequalities and vector calculus.

Problems 3.3 and 3.4 illustrate that L^∞-spaces in one dimension are not subspaces of L^2, but *are* subspaces of H^1.

Problems 3.5 and 3.6 show how you can use the variational formulation of a problem to derive a PDE with boundary conditions, essentially going "backwards". The technique is generally:

1. Use some form of partial integration to eliminate all the derivatives from the test function.
2. Choose a test function that is zero on the appropriate parts of the boundary to eliminate boundary terms from the variational formulation. The remaining terms yield the PDE.
3. Return to the full variational formulation and cancel the PDE found in the previous step. The remaining terms will now yield the boundary condition.

Problem 3.1 An Estimate for the L^2-Norm

Most important results about Sobolev spaces that we are going to use in the design and analysis of numerical methods for partial differential equations concern inequalities connecting relevant Sobolev norms; recall the advice "Do not be afraid of Sobolev spaces! It is only the norms that matter for us, the spaces are irrelvant!" from [NPDE, Sect. 2.2]. An important example of such a fundamental relationship between norms of Sobolev spaces are the Poincaré-Friedrichs inequalities of Theorems [NPDE, Thm. 2.2.18] and [NPDE, Thm. 2.9.6]. In this problem we will prove an estimate related to a Poincaré-Friedrichs inequality in 1D. As a preparation please study the proofs of [NPDE, Thm. 2.2.18] and [NPDE, Thm. 2.9.6].

For $u \in C^1([a,b])$ prove the estimate

$$\int_a^b |u(x)|^2 \,\mathrm{d}x \leq \frac{2}{b-a} \left| \int_a^b u(y) \,\mathrm{d}y \right|^2 + 2(b-a)^2 \int_a^b |u'(t)|^2 \,\mathrm{d}t.$$

Remark. In the theory of Sobolev spaces, usually it is sufficient to prove an inequality between norms for smooth functions only, using classical calculus. The reason is that those functions are dense in the Sobolev spaces, *cf.* the remark in the proof of [NPDE, Thm. 2.2.18].

HINT: Use

1. the fundamental theorem of calculus [NPDE, Eq. (2.4.1)] in the form $u(x) = u(y) + \int_y^x u'(t) \,\mathrm{d}t$,

2. the simple estimate $(a+b)^2 \leq 2(a^2+b^2)$, $a, b \in \mathbb{R}$,

3. and the Cauchy-Schwartz inequality for integrals in 1D, *cf.* [NPDE, Eq. (2.2.17)].

Problem 3.2 Green's Formula (Core problem)

To extract a boundary value problem for a partial differential equations from the variational problem [NPDE, Eq. (2.3.3)] we needed multi-dimensional integration by parts as expressed through Green's first formula of [NPDE, Thm. 2.4.7].

Now prove Greens's formula for $\Omega \subset \mathbb{R}^2$

$$-\int_{\Omega} \operatorname{div} \mathbf{j}\, v \mathrm{d}\mathbf{x} = -\int_{\partial\Omega} \mathbf{j} \cdot \mathbf{n}\, v \,\mathrm{d}S + \int_{\Omega} \mathbf{j} \cdot \mathbf{grad}\; v \,\mathrm{d}\mathbf{x},$$

where $\mathbf{j} \in \mathcal{C}^1(\overline{\Omega})^2$ and $v \in \mathcal{C}^1(\overline{\Omega})$.

HINT: Use Gauss' theorem [NPDE, Thm. 2.4.5].

$$\int_{\Omega} \operatorname{div} \mathbf{F} \,\mathrm{d}\mathbf{x} = \int_{\partial\Omega} \mathbf{F} \cdot \mathbf{n} \,\mathrm{d}S,$$

where $\mathbf{F} \in \mathcal{C}^1(\overline{\Omega})^2$.

Problem 3.3 L^∞-Norms Are Not Bounded by L^2-Norms on $\mathcal{C}^0\big([0,1]\big)$.

This problem strikingly demonstrates that a small L^2-norm of a discretization error need not mean that the error is small locally.

Show that for any $n \in \mathbb{N}$, there is a function $u_n \in \mathcal{C}^0([0,1])$ such that

$$\|u_n\|_{L^2([0,1])} \leq \frac{1}{n}, \quad \text{but} \quad \|u_n\|_{L^\infty([0,1])} \geq n,$$

see [NPDE, Def. 1.6.6] and [NPDE, Eq. (1.6.5)].

HINT: For large n, the function u_n must be large in a very small region ("spike shape").

Problem 3.4 L^∞-Norms *Are* Bounded by H^1-Seminorms in 1D

Show that for any $u \in \mathcal{C}_0^1\big(]0,1[\big)$,

$$\|u\|_{L^\infty(]0,1[)} \leq \|u\|_{H^1(]0,1[)}\,,$$

see [NPDE, Eq. (1.6.5)] and [NPDE, Def. 2.2.12].

Remark 1. By a density argument, see the explanations in the proof of [NPDE, Thm. 2.2.18], we can even conclude that the estimate holds for all functions in $H^1(]0,1[)$.

Remark 2. The above estimate confirms that point evaluation $v \mapsto v(x)]$, $0 < x < 1$, is a *continuous* functional on $H^0(), 1[]1$. Therefore, this is a valid right hand side for a variational problem posed on $H^0(), 1[]1$. This is not the case in higher dimensions as was demonstrated in [NPDE, Ex. 2.3.14].

HINT: Study the proof of [NPDE, Thm. 2.2.18] in 1D. Apply the fundamental theorem of calculus [NPDE, Eq. (2.4.1)].

Problem 3.5 Heat Conduction with Non-Local Boundary Conditions (Core problem)

This problem is meant to practice the conversion of a variational problem into a boundary value for a partial differential equation, see [NPDE, Sect. 2.4] and the extraction of boundary conditions hidden in the variational formulation as in [NPDE, Rem. 2.4.16].

Concretely, we consider the modelling of a two-dimensional cross-section of a submerged insulated wire, see figure 3.1. The wire has a central core of conducting material, say copper, which carries a current. Ohmic losses lead to heat generation in the copper. Copper conducts heat very well and, thus, the copper core can be assumed to have a uniform but unknown temperature.

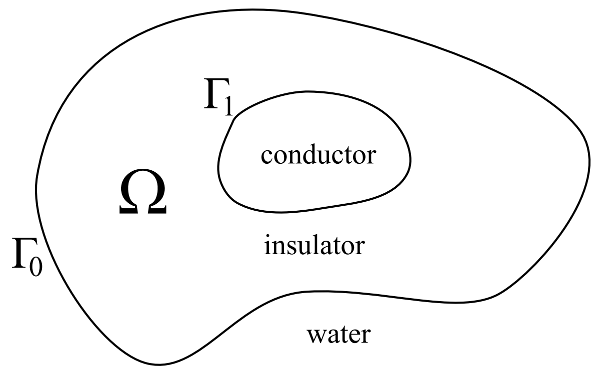

Figure 3.1: Cross-section of a submerged wire.

The copper is surrounded by an annulus of insulator, some plastic, for example, which is again surrounded by water, which we assume to be at a constant temperature of 0. We seek a mathematical model providing us with the temperature distribution within the insulation. Such a model is given by the variational problem

$$u \in V_0 : \int_\Omega \kappa(\mathbf{x}) \operatorname{grad} u(\mathbf{x}) \cdot \operatorname{grad} v(\mathbf{x}) \mathrm{d}\mathbf{x} = \int_{\Gamma_1} v(\mathbf{x}) \mathrm{d}S \,, \quad \forall v \in V_0, \tag{3.5.1}$$

where the heat conductivity κ is uniformly positive and bounded, and with

$$V_0 = \left\{ v \in H^1(\Omega) \,\middle|\, v|_{\Gamma_0} = 0,\, v|_{\Gamma_1} = \text{const} \right\}.$$

(3.5a) Determine a bilinear form a and linear form ℓ so that (3.5.1) becomes an abstract linear variational problem $\mathsf{a}(u, v) = \ell(v)$.

(3.5b) Show that ℓ is continuous with respect to the energy norm induced by a. In the lecture we found this to be an essential condition for the well-posedness of a linear variational problem.

HINT: The energy norm is defined as in [NPDE, Def. 2.1.27], and ℓ must satisfy [NPDE, Eq. (2.2.1)] to be continuous with respect to this norm. Then use the *trace theorem* [NPDE, Thm. 2.9.7].

(3.5c) If u solves (3.5.1) and is sufficiently smooth, it also satisfies a partial differential equation on Ω. Find this equation.

HINT: Follow the approach of [NPDE, Sect. 2.4]: as test functions v use functions in $\mathcal{C}_0^1(\Omega)$, that is, they should be zero on both boundaries Γ_0, Γ_1. Use [NPDE, Thm. 2.4.7] (with $\operatorname{grad} u$ in place of j). Argue what happens to the boundary terms. Then appeal to [NPDE, Lem. 2.4.10].

(3.5d) The function u from problem (3.5c) must also satisfy a certain non-local boundary condition implied by (3.5.1). Find this boundary condition.

HINT: Follow the strategy from [NPDE, Rem. 2.4.16] and use the PDE derived in the previous sub-problem.

(3.5e) What is the physical interpretation of the boundary condition from (3.5d) in terms of heat conduction?

Problem 3.6 Minimization of a Quadratic Functional (Core problem)

[NPDE, Sect. 2.1.3] introduced abstract quadratic minimization problems, see [NPDE, Def. 2.1.17] and [NPDE, Def. 2.1.21]. As concrete examples arising from equilibrium models we studied quadratic minimization problems posed on the Sobolev spaces $H_0^1(\Omega)$ and $H^1(\Omega)$ of scalar functions. In [NPDE, Sect. 2.3], we learned how to convert a quadratic minimization problem into variational form, see [NPDE, Eq. (2.3.7)]. [NPDE, Sect. 2.4] taught us how to use multidimensional integration by parts [NPDE, Thm. 2.4.7] to convert the linear variational problems on Sobolev spaces into a boundary value problems for a 2nd-order elliptic PDEs. In this exercise we practise all these steps in the case of an "exotic" quadratic minimization problem.

We consider the quadratic functional

$$J(\mathbf{u}) = \int_\Omega \left|\operatorname{div}\mathbf{u}(\mathbf{x})\right|^2 + \|\mathbf{u}(\mathbf{x})\|^2 + \mathbf{f}(\mathbf{x})\cdot\mathbf{u}(\mathbf{x})\mathrm{d}\mathbf{x}\;, \tag{3.6.1}$$

with $\Omega \subset \mathbb{R}^3$ bounded, and for functions $\mathbf{u} : \Omega \to \mathbb{R}^3$, that is, J takes *vector field arguments*.

(3.6a) Identify the bilinear form a and the linear form ℓ in the quadratic functional J, *cf.* [NPDE, Def. 2.1.17].

HINT: See [NPDE, Def. 2.1.17].

(3.6b) Show that the bilinear form a from subproblem (3.6a) is symmetric and positive definite, see [NPDE, Def. 2.1.25].

HINT: See [NPDE, Eq. (2.1.19)] and [NPDE, Def. 2.1.25].

(3.6c) Show that the linear form ℓ from subproblem (3.6a) is continuous with respect to the energy norm induced by a.

HINT: The energy norm is defined as in [NPDE, Def. 2.1.27], and ℓ must satisfy [NPDE, Eq. (2.2.1)] to be continuous with respect to this norm.

(3.6d) Explain why the Sobolev space

$$H(\operatorname{div},\Omega) := \left\{\mathbf{v} : \Omega \to \mathbb{R}^3 \text{ integrable} \;\middle|\; \int_\Omega \left|\operatorname{div}\,\mathbf{v}\right|^2 + \|\mathbf{v}\|^2\,\mathrm{d}\mathbf{x} < \infty\right\}.$$

provides the right framework for studying the minimization problem for the functional J from (3.6.1).

(3.6e) Derive and state the linear variational problem equivalent to the minimization problem

$$\mathbf{u}_* = \underset{\mathbf{v} \in H(\mathrm{div}, \Omega)}{\operatorname{argmin}} J(\mathbf{v}).$$

HINT: See [NPDE, Eq. (2.3.6)] and [NPDE, Eq. (2.3.7)].

(3.6f) Derive the *partial differential equation* on Ω that arises from the variational problem from (3.6e).

HINT: Follow the approach of [NPDE, Sect. 2.4], in particular [NPDE, Rem. 2.4.16]: as test functions $\mathbf{v}$ use vector fields in $(\mathcal{C}_0^1(\Omega))^3$, that is, they should be zero on the boundary. Use [NPDE, Thm. 2.4.7] (with $\mathrm{div}\ \mathbf{u}$ in place of v and $\mathbf{v}$ in place of $\mathbf{j}$) in order to "shift the div from $\mathbf{v}$ onto $\mathrm{div}\, u$ as $-\mathrm{grad}$". Argue, what happens to the boundary terms. Then appeal to [NPDE, Lem. 2.4.10].

(3.6g) The variational problem from (3.6e) also implies boundary conditions. Which?

HINT: Follow the strategy from [NPDE, Rem. 2.4.16] and use the PDE derived in subproblem (3.6f).

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References

[NPDE] Lecture Slides for the course "Numerical Methods for Partial Differential Equations", SVN revision # 53384.

[NCSE] Lecture Slides for the course "Numerical Methods for CSE".

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