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Numerical Methods for Partial Differential Equations

ETH Zürich
D-MATH

Homework Problem Sheet 8

Introduction. This problem sheet mainly deals with issues related to [NPDE, Sect. 5.2] and [NPDE, Sect. 5.3].

Problem 8.1 Shape Regularity and Angle Condition

When deriving interpolation error estimates for linear interpolation on triangular meshes in [NPDE, Sect. 5.3.2], it was convenient to introduce the concept of a shape regularity measure ρ_K for a triangle K, see [NPDE, Def. 5.3.26]. The intuition is that in 2D the shape regularity measure indicates the degree to which a triangle is distorted. This distortion was linked to the angles in [NPDE, Fig. 157], [NPDE, Fig. 158], and [NPDE, Fig. 159]. This link will be explored in this problem.

Bound the smallest angle of a triangle K by an expression involving only ρ_K.

HINT: From secondary school recall the formula $|K| = \frac{1}{2}ab\sin(\gamma)$, where γ is the angle enclosed by the sides with lengths a, b.

Problem 8.2 Localized Interpolation Error Estimates

There is a more refined way than that of [NPDE, Thm. 5.3.27] to state interpolation error estimates. It relies on the piecewise constant *meshwidth function*

$$\hbar(\boldsymbol{x}) = h_K \quad \text{if} \quad \boldsymbol{x} \in K, \tag{8.2.1}$$

where K is a cell of a triangular mesh $\mathcal{M}$ of a domain $\Omega \subset \mathbb{R}^2$.

Based on [NPDE, Lem. 5.3.25] derive the estimate

$$\left\| \hbar^{-2}(u - \mathsf{I}_1 u) \right\|_{L^2(\Omega)} \leq C |u|_{H^2(\Omega)} \quad \forall u \in H^2(\Omega), \tag{8.2.2}$$

What is a concrete value for the constant C?

Problem 8.3 Projection onto Constants (Core problem)

In [NPDE, Sect. 5.3.1] we derived L^2- and H^1-estimates for the error of piecewise linear interpolation on a grid, see [NPDE, Eq. (5.3.10)] and [NPDE, Eq. (5.3.12)]. The key tool was the integral representation formula [NPDE, Eq. (5.3.7)]. In this problem we practice these techniques for an even simpler projection operator.

Given a grid $\mathcal{M} := \{]x_{j-1}, x_j[\colon 1 \leq j \leq M\}$ of $[a, b] \subset \mathbb{R}$ we define a projection onto the space

$\mathcal{S}_0^{-1}(\mathcal{M})$ of piecewise constant discontinuous functions on $\mathcal{M}$ according to

$$\mathsf{I}_0 : \begin{cases} L^2(]a,b[) & \mapsto \mathcal{S}_0^{-1}(\mathcal{M}) \\ u & \mapsto \sum_{j=1}^{M} \frac{1}{|x_j - x_{j-1}|} \int_{x_{j-1}}^{x_j} u(\xi)\,\mathrm{d}\xi \cdot \chi_{]x_{j-1},x_j[}, \end{cases} \tag{8.3.1}$$

where $\chi_{]x_{j-1},x_j[}$ stands for the characteristic function of the interval $]x_{j-1},x_j[$, that is

$$\chi_{]x_{j-1},x_j[}(x) = \begin{cases} 1 & \text{, if } x \in]x_{j-1},x_j[, \\ 0 & \text{else.} \end{cases} \tag{8.3.2}$$

We abbreviate $K :=]x_{j-1},x_j[$ for some $j = 1,\dots,M$.

Remark. The linear projection I_0 is an instance of an L^2-projection. Generically, given a (closed) subspace $V \subset L^2(\Omega)$, the associated L^2-projection operator $\mathsf{Q}_V : L^2(\Omega) \mapsto V$ is defined through as solution operator of the variational problem

$$\mathsf{Q}_V u \in V : \quad (\mathsf{Q}_V u, v)_{L^2(\Omega)} = (u,v)_{L^2(\Omega)} \quad \forall v \in V\ .$$

(8.3a) Compute $\mathsf{I}_0 u$ on $[0,1]$ for $u(x) = x$ and an equidistant mesh with meshwidth $h := M^{-1}$. Sketch the function $\mathsf{I}_0 u$.

(8.3b) Derive the local integral representation formula for the projection error

$$(u - \mathsf{I}_0 u)(x) = \frac{1}{|x_j - x_{j-1}|} \int_{x_{j-1}}^{x_j} \int_y^x u'(\xi)\,\mathrm{d}\xi\mathrm{d}y, \quad x_{j-1} < x < x_j. \tag{8.3.3}$$

HINT: Use the fundamental theorem of calculus [NPDE, Eq. (2.4.1)].

(8.3c) Starting from (8.3.3) deduce the estimate

$$\|u - \mathsf{I}_0 u\|^2_{L^2(]x_{j-1},x_j[)} \le |x_j - x_{j-1}|^2 |u|^2_{H^1(]x_{j-1},x_j[)}. \tag{8.3.4}$$

HINT: Apply the Cauchy-Schwarz inequality for integrals [NPDE, Eq. (2.2.17)] twice.

(8.3d) Based on (8.3.4) derive the global projection error estimate

$$\|u - \mathsf{I}_0 u\|_{L^2(]a,b[)} \le h_{\mathcal{M}} |u|_{H^1(]a,b[)}, \tag{8.3.5}$$

where $h_{\mathcal{M}}$ is the meshwidth of $\mathcal{M}$.

Problem 8.4 An Impossible Interpolation Estimate

[NPDE, Thm. 5.3.27] gave us bounds for the $L^2(\Omega)$-norm and $H^1(\Omega)$-seminorm of the error of piecewise linear interpolation on a triangular mesh of $\Omega \subset \mathbb{R}^2$. These bounds invariably contained the $H^2(\Omega)$-norm of the interpolated function. Now, somebody claims to have found an analogous interpolation estimate of the form

$$\|u - \mathsf{I}_1 u\|_{L^2(\Omega)} \le C h_{\mathcal{M}} \rho_{\mathcal{M}} |u|_{H^1(\Omega)} \quad \forall u \in H^1(\Omega), \tag{8.4.1}$$

with some constant $C > 0$.

(8.4a) Show that (8.4.1) implies

$$\|\mathsf{I}_1 u\|_{L^2(\Omega)} \leq C\|u\|_{H^1(\Omega)} \quad \forall u \in H^1(\Omega), \tag{8.4.2}$$

with a constant $C > 0$ whose dependence of $h_{\mathcal{M}}$ and $\rho_{\mathcal{M}}$ should be made explicit.

HINT: First study [NPDE, Rem. 5.3.33].

(8.4b) Argue why (8.4.2) cannot be true.

HINT: Remember [NPDE, Rem. 2.3.14], [NPDE, Cor. 2.3.19]. Note that we are in a 2D setting.

Problem 8.5 Experience Interpolation Errors (Core problem)

The experimental convergence studies presented in [NPDE, Sect. 5.2] relied on LehrFEM codes like [NPDE, Code 5.2.3]. In this problem you are supposed to perform a simple computation of an interpolation error using the facilities of LehrFEM.

This problem introduces to the "practice of linear interpolation":

(8.5a) Based on the MATLAB LehrFEM library write a MATLAB function

```
L2err = L2Iptr(Mesh,u)
```

meant to approximately compute the interpolation error $\|u - \mathsf{I}_1 u\|_{L^2(\Omega)}$ for piecewise linear interpolation of u (passed through the function handle `u` of type `@(x,varargin)`) on a mesh passed in the LehrFEM mesh data structure `Mesh`.

HINT: For the evaluation of local integrals use the quadrature routine `P7O6` from the MATLAB library [LehrFEM, Sect. 3.3.2]. Use LehrFEM functions whenever possible, see [NPDE, Rem. 5.2.3].

HINT: The reference implementation `L2Iptr_ref` expects the handle `u` to be *vector safe*, that is, it can accept an $N \times 2$-matrix `x` of several points at once.

(8.5b) Test your implementation: apply `L2Iptr(Mesh,u)` to $u(\boldsymbol{x}) = \sin(\|\boldsymbol{x}\|)$ on $\Omega = (0,1)^2$ using the mesh $\mathcal{M}_0$ `Coordinates_Mzero.dat` and `Elements_Mzero.dat`

HINT: For this case the $L^2(\Omega)$-norm of the interpolation error is 0.0172.

(8.5c) For the function u from subproblem (8.5b) compute $\|u - \mathsf{I}_1 u\|_{L^2(\Omega)}$ for a sequence of meshes $\mathcal{M}_0, \mathcal{M}_1, \ldots, \mathcal{M}_5$ created by successive regular refinement of $\mathcal{M}_0$. Plot this interpolation error norm vs. $h_{\mathcal{M}_k}$ in doubly logarithmic scale.

HINT: Regular refinement of a triangular mesh is done by the `refine_REG` function as explained in [LehrFEM, Sect. 1.4.1].

Listing 8.1: Testcalls for Problem 8.5

```
tm.Coordinates = [0 0; 1 0; 1 1; 0 1];
tm.Elements = [1 2 4; 2 3 4];
tm = add_Edges(tm);
tm.ElemFlag = ones(size(tm.Elements,1),1);
loc = get_BdEdges(tm);
```

```
tm.BdFlags = zeros(size(tm.Edges,1),1);
tm.BdFlags(loc) = -1;

u = @(x,varargin) x(:,1).^2 .* x(:,2).^2;

[L2e, L2n, hm] = L2Itpr(tm, u);
fprintf('## L2Itpr\n');
[L2e, L2n, hm]
```

Listing 8.2: Output for Testcalls for Problem 8.5

```
## L2Itpr

ans =

    0.1035    0.2887    1.4142
```

Problem 8.6 Convergence of Finite Element Solutions

A student is testing his implementation of a finite element method. On the square domain $\Omega = (0,1)^2$ he considers the 2nd-order elliptic boundary value problem

$$\begin{aligned} -\Delta u &= 1 && \text{in } \Omega, \\ u &= \frac{1}{4}\left(1 - \|\boldsymbol{x}\|^2\right) && \text{on } \partial\Omega. \end{aligned} \tag{8.6.1}$$

He computes an approximate solutions $u_N \in \mathcal{S}_p^0(\mathcal{M})$ by means of a finite element Galerkin method using linear ($p = 1$) and quadratic ($p = 2$) Lagrangian finite elements on a sequence of triangular meshes $\mathcal{M}$.

The following table lists the measured $H^1(\Omega)$-seminorm of the discretization error as a function of the meshwidth $h_{\mathcal{M}}$.

$h_{\mathcal{M}}$	0.70	0.35	0.17	0.088	0.044	0.022	0.011
$\mathcal{S}_1^0(\mathcal{M})$	0.10	0.051	0.025	0.012	0.0064	0.0032	0.0008
$\mathcal{S}_2^0(\mathcal{M})$	$1.75{\cdot}10^{-16}$	$1.24{\cdot}10^{-15}$	$5.71{\cdot}10^{-15}$	$2.29{\cdot}10^{-14}$	$8.91{\cdot}10^{-14}$	$3.53{\cdot}10^{-13}$	$1.41{\cdot}10^{-12}$

The data of this table are available in the MATLAB data file `cvgtab.mat`.

(8.6a) Show that $u(\boldsymbol{x}) = \frac{1}{4}(1 - \|\boldsymbol{x}\|^2)$ is the exact solution of (8.6.1)

(8.6b) What kind of convergence (qualitative and quantitative) for linear Lagragian finite elements can be inferred from the error table?

(8.6c) Explain the striking difference between the behavior of the discretization error for linear and quadratic Lagrangian finite elements.

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To be submitted on April 29. **MATLAB:** Submit all files in the online system.

Include the files that generate the plots. Label all your plots. Include commands to run your functions. Comment on your results.

References

[NPDE] Lecture Slides for the course "Numerical Methods for Partial Differential Equations", SVN revision # 54427.

[NCSE] Lecture Slides for the course "Numerical Methods for CSE".

[LehrFEM] LehrFEM manual.

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