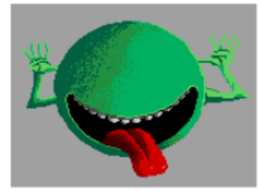
Course 401-0674-00L: NPDE
Endterm, 31-05-2013
Prof. Ralf Hiptmair



Don't panic!
Good luck!

Name
Student number

Points

Problem 0.1 Discretization error for linear and quadratic Lagrangian finite elements [5 points]

On a polygonal, bounded domain $\Omega \subset \mathbb{R}^2$ we consider the finite element Galerkin discretization of the boundary value problem

$$-\Delta u + u = f \in L^2(\Omega) \quad \text{in } \Omega \subset \mathbb{R}^2, \quad u = 0 \quad \text{on } \partial\Omega\,. \tag{0.1.1}$$

by means of piecewise linear Lagrangian finite elements (FE space $\mathcal{S}^0_{1,0}(\mathcal{M})$) and piecewise quadratic Lagrangian finite elements (FE space $\mathcal{S}^0_{2,0}(\mathcal{M})$) on a triangular mesh $\mathcal{M}$. The respective finite element solutions will be denoted by $u_L \in \mathcal{S}^0_{1,0}(\mathcal{M})$ and $u_Q \in \mathcal{S}^0_{2,0}(\mathcal{M})$.

(0.1a) **[3 points]** Show that

$$\|u - u_Q\|_{\mathsf{a}}^2 + \|u_Q - u_L\|_{\mathsf{a}}^2 = \|u - u_L\|_{\mathsf{a}}^2\,, \tag{0.1.2}$$

$u \in H^1_0(\Omega)$ is the exact solution and $\|\cdot\|_{\mathsf{a}}$ stands for the energy norm induced by the variational formulation of (0.1.1).

Solution: Owing to the embedding $\mathcal{S}^0_1(\mathcal{M}) \subset \mathcal{S}^0_2(\mathcal{M})$ we have Galerkin orthogonality $\mathsf{a}(u - u_Q, u_Q - u_L) = 0$. Then use Pythagoras' theorem.

(0.1b) **[2 points]** Give an argument, why

$$\|u - u_Q\|_{\mathsf{a}} \leq \|u - u_L\|_{\mathsf{a}} \tag{0.1.3}$$

holds true.

Solution: This is a trivial consequence of (0.1.2). Equivalently, one may appeal to the optimality of the Galerkin solution with respect to the energy norm and the embedding $\mathcal{S}^0_1(\mathcal{M}) \subset \mathcal{S}^0_2(\mathcal{M})$.

Problem 0.2 Convergence of finite element solutions [6 points]

On the "L-shaped" domain $\Omega =]-1,1[^2 \setminus [-1,0]^2$ we consider the second-order elliptic boundary value problem

$$-\Delta u = f \quad \text{in } \Omega \quad , \quad u = g \quad \text{on } \partial\Omega\,. \tag{0.2.1}$$

In a code a Galerkin discretization by means of piecewise linear and quadratic Lagrangian finite elements is employed.

(0.2a) [3 points] Consider the case when f and g are set to produce the exact solution $u(\boldsymbol{x}) = \cos(\pi x_1)\cos(\pi x_2)$.
Describe in qualitative and quantitative terms the convergence of the finite element solutions in the energy norm on a sequence of triangular meshes created by successive regular refinement of some initial mesh.

Solution: Note that in this case the solution u is smooth despite the presence of a re-entrant corner at $\boldsymbol{x} = 0$! In particular we have $u \in H^3(\Omega)$. The energy norm for the boundary value problem agrees with $|\cdot|_{H^1(\Omega)}$.

Also observe that all meshes in the sequence enjoy the same shape-regularity measure. Therefore, from [**?**, Thm. 5.3.40] we conclude *algebraic convergence* with the following rates

$$\begin{aligned} &\text{for } V_{0.N} = \mathcal{S}_1^0(\mathcal{M}): \quad \|u - u_N\|_{\mathsf{a}} \le C h_{\mathcal{M}}\,, \\ &\text{for } V_{0,N} = \mathcal{S}_2^0(\mathcal{M}): \quad \|u - u_N\|_{\mathsf{a}} \le C h_{\mathcal{M}}^2\,, \end{aligned}$$

where $h_{\mathcal{M}}$ is the mesh width and $C > 0$ is independent of $\mathcal{M}$.

(0.2b) [3 points] Somebody else uses the code on the boundary value problem (0.2.1) for $f \equiv 1$ and $g = 0$ and he observes the errors in energy norm displayed in Figure 0.1 for the finite element solutions on a sequence of triangular meshes created by successive regular refinement of some initial mesh.

Explain, why the answer to sub-problem (0.2a) completely fails to match the observations in this case.

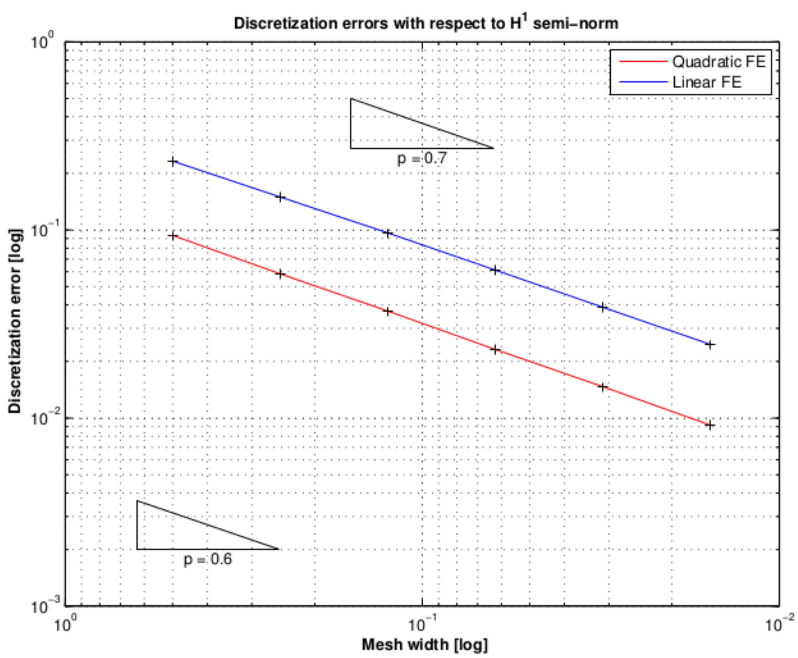


Figure 0.1: Energy norm of discretization errors for both linear and quadratic Lagrangian finite elements.

Solution: The gradient of the solution is singular at the origin so that the solution u does not even belong to $H^2(\Omega)$. Thus piecewise quadratic approximation does not yield an improved rate of (algebraic) convergence compared to piecewise linear approximation. If $u \notin H^2(\Omega)$ we have no guarantee even for $O(h)$ convergence, which is obviously not achieved in the experiment.

Problem 0.3 Linear output functionals [6 points]

Which of the following output functionals are linear and well defined on $L^2(\Omega)$ and $H^1(\Omega)$, respectively, for $\Omega = \{\boldsymbol{x} \in \mathbb{R}^2 : \|\boldsymbol{x}\| < 1\}$? Answer by entering "YES" or "NO" in the blank fields of the table.

functional	linear?	defined on $L^2(\Omega)$?	defined on $H^1(\Omega)$?
$J(v) = \int_\Omega \mathbf{c} \cdot \mathbf{grad}\, v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x},\ \mathbf{c} \in \mathbb{R}^2$	YES	NO	YES
$J(v) := \int_{\partial\Omega} \mathbf{grad}\, v(\boldsymbol{x}) \cdot \boldsymbol{n}(\boldsymbol{x})\,\mathrm{d}S(\boldsymbol{x})$	YES	NO	NO
$J(v) := \lvert v(\boldsymbol{x}_0)\rvert,\ \boldsymbol{x}_0 \in \Omega$	NO	NO	NO
$J(v) := \int_\Omega \mathbf{c} v(\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|})\,\mathrm{d}\boldsymbol{x},\ \mathbf{c} \in \mathbb{R}^2$	YES	NO	YES

Problem 0.4 Parabolic evolution [5 points]

For testing purposes one considers the parabolic evolution problem

$$\begin{aligned} \frac{\partial u}{\partial t} - \Delta u = 0 \quad & \text{in } \Omega \times]0,T[\ , \\ u = 0 \quad & \text{on } \partial\Omega \times]0,T[\ , \\ u(\boldsymbol{x},0) = u_0(\boldsymbol{x}) \quad & \text{for } \boldsymbol{x} \in \Omega\ , \end{aligned} \tag{0.4.1}$$

on the unit square $\Omega =]0,1[^2$. Choosing $u_0(\boldsymbol{x}) = \sin(\pi x_1)\sin(\pi x_2)$ one obtains $u(\boldsymbol{x},t) = \exp(-\pi^2 t) u_0(\boldsymbol{x})$ as exact solution.

A method of lines approach is employed: Discretization in space relies on quadratic Lagrangian finite elements, whereas discretization in time is done using an L-stable SDIRK implicit Runge-Kutta scheme of order 2 with uniform timestep $\tau > 0$.

(0.4a) [3 points] For fixed timestep τ we examine the $L^2(\Omega)$-norm of the discretization error at final time $T = \frac{1}{2}$ for an (infinite) sequence of meshes created by uniform regular refinement. Indicate the qualitative dependence of this error norm on the mesh-width h by drawing a suitable error curve in Figure 0.2.

HINT: Assume an error norm of 1 on the coarsest mesh.

Solution: Since u is smooth we expect initial algebraic convergence $O(h^3)$, which however, will level off, as the temporal discretization error becomes dominant.

(0.4b) [2 points] Now we track the error norm $E(t_j) := \|u(t_j) - u_N(t_j)\|_{L^2(\Omega)}$ as a function of $t_j = j\tau$, $j \in \mathbb{N}$, for fixed finite element mesh and fixed timestep τ. What can we expect? Sketch E in Figure 0.3, assuming $E(0) = 0.2$.

Solution: We expect a geometric decay of the error, since the norm of the solution $u(T)$ will also

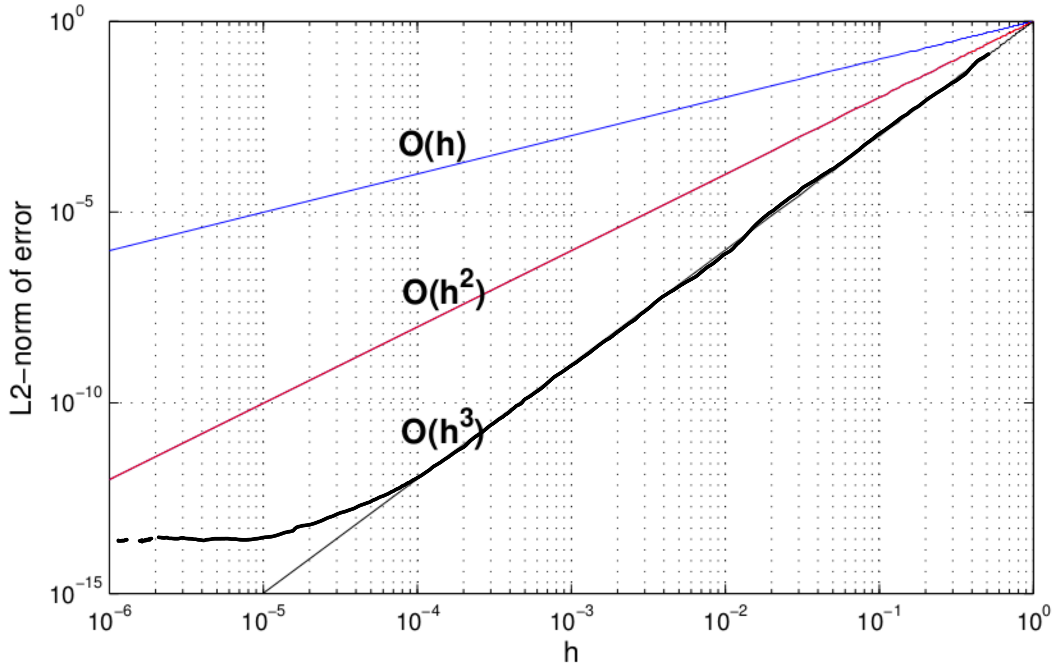


Figure 0.2: Empty double logarithmic coordinate system, mesh-width h versus $\|u(T) - u_N(T)\|_{L^2(\Omega)}$, $T > 0$ fixed.

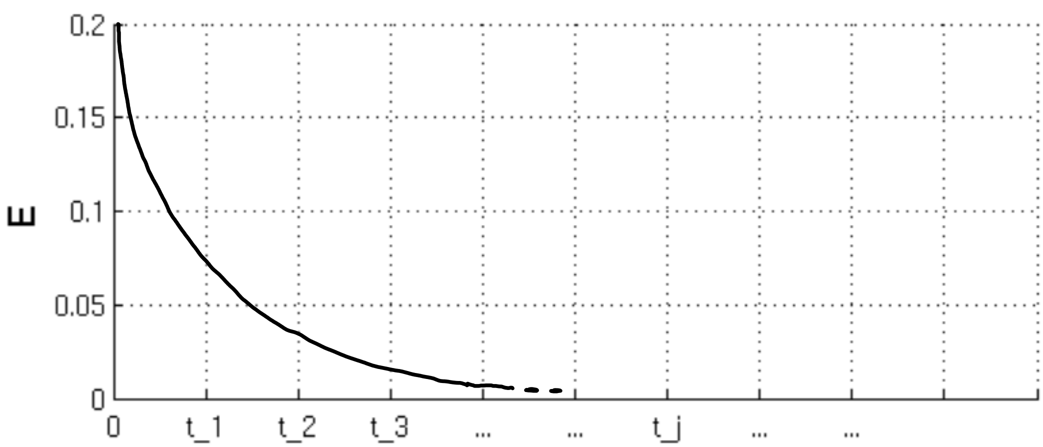


Figure 0.3: Empty linear coordinate system discrete times $t_1, t_2, ...t_j, ...$ vs. E. Timestep τ and mesh fixed.

decay exponentially. The rate of this geometric decay may not be the same as the rate with which $u(T)$ tends to zero.

Problem 0.5 Singular perturbations [3 points]

Explain the concept of singular perturbation of a boundary value problem for the BVP

$$-\epsilon\Delta u + \mathbf{v} \cdot \mathbf{grad}\, u = 0 \quad \text{in } \Omega \quad , \quad u = g \quad \text{on } \partial\Omega \, , \tag{0.5.1}$$

as $\epsilon \to 0$. Here Ω is a domain in $\mathbb{R}^2$ and $\mathbf{v} \in \mathbb{R}^2 \setminus \{0\}$.

Solution: In the limit case $\epsilon = 0$ Dirichlet boundary conditions at the outflow boundary can no longer be satisfied, which manifests itself in the emergence of *boundary layers* for $\epsilon \ll 1$.