Lecture Notes for the Algebraic Geometry course held by Rahul Pandharipande

Endrit Fejzullahu, Nikolas Kuhn, Vlad Margarint,
Nicolas Müller, Samuel Stark, Lazar Todorovic

July 28, 2014

Contents

0 References

Good references for the topics here:

1. Harris, Algebraic Geometry, Springer.
2. Hartshorne, Algebraic Geometry, Springer. Chapter I, but also beginning of Chapter II for schemes.
3. Shafarevich, Basic algebraic geometry I, II.
4. Atiyah, Macdonald Commutative Algebra (for basic commutative algebra).

1 Affine varieties

In this course we mainly consider algebraic varieties and schemes. It is worth noting that several definitions related to algebraic varieties are formally similar to those involving C^∞-manifolds. (In algebraic geometry the local analysis of algebraic varieties is by commutative algebra, whereas in differential geometry the local analysis of C^∞-manifolds is by calculus.) In the following we take the complex field $\mathbb{C}$ to be the underlying field (one could also consider any algebraically closed field instead; the discussion would be identical).

By *affine n-space* over $\mathbb{C}$ we mean $\mathbb{C}^n$. For a subset X of $\mathbb{C}[x_1, \ldots, x_n]$ we denote by $V(X) := \{x \in \mathbb{C}^n \mid f(x) = 0 \text{ for all } f \in X\}$ the common zero locus of the elements of X. Notice that $V(X) = V(\langle X \rangle)$. A subset of $\mathbb{C}^n$ of the form $V(X)$ for some subset X of $\mathbb{C}[x_1, \ldots, x_n]$ is said to be an *affine algebraic variety*. By Hilbert's basis theorem $\mathbb{C}[x_1, \ldots, x_n]$ is Noetherian, so that every affine algebraic variety is in fact the common zero locus of finitely many polynomials. For a subset S of $\mathbb{C}^n$ the *ideal* of S is the set of polynomials vanishing on S, $\mathcal{I}(S) := \{f \in \mathbb{C}[x_1, \ldots, x_n] \mid f(x) = 0 \text{ for all } x \in S\}$.

Both $\mathcal{I}$ and V are inclusion-reversing, and are connected by Hilbert's Nullstellensatz: for every ideal I of $\mathbb{C}[x_1, \ldots, x_n]$ we have $\mathcal{I}(V(I)) = \sqrt{I}$. Hence $\mathcal{I}$ and V induce a bijective inclusion-reversing correspondence between affine algebraic varieties and radical ideals of $\mathbb{C}[x_1, \ldots, x_n]$.

The *Zariski topology* on $\mathbb{C}^n$ is defined by letting the closed subsets of $\mathbb{C}^n$ be the affine algebraic varieties. The Zariski topology on a variety V in $\mathbb{C}^n$ is the induced (subspace) topology; a more intrinsic definition relies on the following notion: a *subvariety* W of V is a variety W in $\mathbb{C}^n$ such that $W \subset V$. Then a subset U of V is open in the Zariski topology of V if and only if $V \setminus U$ is a subvariety of V. Let us now consider the sets $\{U_f\}_{f \in \mathbb{C}[x_1, \ldots, x_n]}$ defined by $U_f := \{x \in V \mid f(x) \neq 0\}$.

Remark 1.1 $\{U_f\}_{f \in \mathbb{C}[x_1, \ldots, x_n]}$ is a basis of the Zariski topology: let $U \subset V$ be an open subset. By definition, $U^c = V(I)$ for some ideal I of $\mathbb{C}[x_1, \ldots, x_n]$. Since $\mathbb{C}[x_1, \ldots, x_n]$ is noetherian, I is finitely generated , i.e. $I = \langle f_1, \ldots, f_k \rangle$ for some polynomials. Therefore we have

$$U = V(I)^c = V(\langle f_1, \ldots, f_k \rangle)^c = (\cap_{i=1}^k V(\langle f_i \rangle))^c = \cup_{i=1}^k U_{f_i},$$

so every open subset is a union of just finitely many basic open subsets.

It follows from this also that $\mathbb{C}^n$ is quasi-compact.

Example 1.2 As every nonzero polynomial $f \in \mathbb{C}[x]$ has at most finitely many zeros, the Zariski topology on the affine line $\mathbb{C}^1$ is exactly the cofinite topology. Notice that in this topology every injection $f : \mathbb{C} \to \mathbb{C}$ is continuous, in contrast to the standard (euclidean) topology.

2 Morphisms of affine varieties

Let $V \subset \mathbb{C}^n$ an affine algebraic variety, $U \subset V$ an open subset.

Definition 2.1 $f : U \to \mathbb{C}$ is an algebraic (regular) function if for each $p \in \mathbb{C}$ there exist an open subset $W_p \subset U$ containing p and $g_p, h_p \in \mathbb{C}[x_1, \ldots, x_n]$ such that

1. $0 \notin g_p(W_p)$.
2. $f|_{W_p} = \frac{h_p}{g_p}$.

Theorem 2.2 A map $f : \mathbb{C}^n \to \mathbb{C}$ is algebraic if and only if f is a polynomial function on $\mathbb{C}^n$.

Proof. The "if" part is trivial.
For the "only if" part, let $f : \mathbb{C}^n \to \mathbb{C}$ be an algebraic map. Then for every $p \in \mathbb{C}$ there is an open neighborhood W_p of p in $\mathbb{C}^n$ and polynomial functions h_p and g_p such that $f|_{W_p} = \frac{h_p}{g_p}$ and g_p never vanishes on W_p. Inside W_p we can find a basic open set $p \in U_{r_p} \subset W_p$, where

$$U_{r_p} = \{\rho \in \mathbb{C}^n \mid r_p(\rho) \neq 0\}.$$

It follows that for all points $p \in \mathbb{C}$ we have a $r_p \in \mathbb{C}[x_1, \ldots, x_n]$ such that $f|_{U_{r_p}} = \frac{h_p}{g_p}$ and $\forall q \in \mathbb{C}^n : g_p(q) = 0 \to r_p(q) = 0$.
With Hilbert's Nullstellensatz it follows that $r_p \in \sqrt{(g_p)}$ and therefore $r_p^k = \alpha_p g_p$ for some $\alpha_p \in \mathbb{C}[x_1, \ldots, x_n]$. On U_{r_p} we have that r_p, g_p, α_p are never 0.
On U_{r_p} we have

$$f|_{U_{r_p}} = \frac{h_p \alpha_p}{g_p \alpha_p} = \frac{h_p \alpha_p}{r_p^k}$$

Define $\hat{h}_p = h_p \alpha_p$ and $\hat{r}_p = r_p^k$. Then $U_{r_p} = U_{r_p^k}$, and for every $p \in \mathbb{C}^n$ there is r_p with $r_p(p) \neq 0$ and $f|_{U_{r_p}} = \frac{h_p}{r_p}$ (removing the hats again).
It follows that finitely many U_{r_p} suffice to cover $\mathbb{C}^n$. Take $p_1, \ldots, p_m$ such that $U_{r_{p_1}}, \ldots, U_{r_{p_m}}$ cover $\mathbb{C}^m$ and write r_i instead of r_{p_i}. We have (because of the cover) $(r_1, \ldots, r_m) = (1)$ and therefore

$$\exists s_1, \ldots, s_m \in \mathbb{C}[x_1, \ldots, x_n] : \sum_{i=1}^m s_i r_i = 1$$

We have $f|_{U_{r_i}} = \frac{h_i}{r_i}$ and $U_{r_i} \cap U_{r_j} = U_{r_i r_j}$. On $U_{r_i r_j}$ we have

$$\frac{h_i}{r_i} = \frac{h_j}{r_j}, h_i r_j = h_j r_i, h_i r_j - h_j r_i = 0 \text{ on } U_{r_i r_j}$$

$$\underbrace{r_i}_{\neq 0} \underbrace{r_j}_{\neq 0} (h_i r_j - h_j r_i) = 0 \text{ everywhere on } \mathbb{C}^n \tag{$*$}$$

$$h_i r_j - h_j r_i = 0 \in \mathbb{C}[x_1, \ldots, x_n]$$

We have to prove $f \in \mathbb{C}[x_1, \ldots, x_n]$. Let

$$F = \sum_{k=1}^m h_k s_k$$

F is a polynomial. We want to show $F = f$ everywhere.

$$r_i F = \sum_{k=1}^m r_i h_k s_k = \sum_{k=1}^m r_k h_i s_k = h_i \sum_{k=1}^m s_k r_k = h_i,$$

because of $(*)$. And on U_{r_i} we have

$$f|_{U_{r_i}} = \frac{h_i}{r_i}, F|_{U_{r_i}} = \frac{h_i}{r_i}$$

□

If $V \subset \mathbb{C}^n$ is an affine algebraic variety and $f : V \to \mathbb{C}$ is an algebraic function, f is restriction of a polynomial (see exercises). If $f_1, f_2 : V \to \mathbb{C}$ are algebraic and the same function $\Leftrightarrow f_1 - f_2 \in \mathcal{I}(V)$. We denote by $\Gamma(V)$ the ring of algebraic functions $V \to \mathbb{C}$.

Proposition 2.3

$$\Gamma(V) \cong \frac{\mathbb{C}[x_1, \ldots, x_n]}{\mathcal{I}(V)}$$

Proof. By

$$f \in \mathbb{C}[x_1, \ldots, x_n] \mapsto f|_V : V \to \mathbb{C}$$

we have

$$\frac{\mathbb{C}[x_1, \ldots, x_n]}{\mathcal{I}(V)} \subset \Gamma(V)$$

So, to prove the proposition, we need to show that this is surjective. Go through the proof of Theorem 2.2 step by step. It is the same, except for one step.
Again, we have $U_{r_i} \cap U_{r_j} = U_{r_i r_j}$. On $U_{r_i r_j}$ we have $\frac{h_i}{r_i} = \frac{h_j}{r_j}$. Let $\hat{h}_i = h_i r_i, \hat{r}_i = r_i^2$. And we have $h_i r_j - r_i h_j = 0$ on $U_{r_i r_j}$. And

$$r_i r_j (h_i r_j - r_i h_j) = 0 \text{ everywhere on } V$$

(see exercise 1.3 on the exercise sheets) □

Example 2.4 $V = (x^2 - y^3) \subset \mathbb{C}^2$. We look at

$$f : V \to \mathbb{C}$$
$$(x, y) \mapsto \begin{cases} \frac{x}{y} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

Let $U = V \setminus \{(0, 0)\}$. $f|_U = \frac{x}{y}$. y is never 0 in U.
Claim: f is not algebraic on V.

Proof. Suppose it were. Then there is a basic open set U_r, $r \in \mathbb{C}[x, y]$, with $(0, 0) \in U_r$, and there are $h, g \in \mathbb{C}[x, y]$ such that

$$\forall (x, y) \in U_r \setminus \{(0, 0)\} : \frac{x}{y} = \frac{h}{g}$$

and g is never 0 on U_r. We have $gx - hy = 0$ on U_r and $r(gx - hy) = 0$ on V. We have

$$x^2 - y^3 | r(gx - hy)$$

because $(x^2 - y^3)$ is a radical ideal. $x^2 - y^3$ can't divide r since $r(0, 0) \neq 0$. Therefore

$$x^2 - y^3 | gx - hy$$

Because $g(0, 0) \neq 0$, g has a nonzero constant term.
Use the ring homomorphism

$$\mathbb{C}[x, y] \to \mathbb{C}[t]$$
$$x \mapsto t^3$$
$$y \mapsto t^2$$

Then $x^2 - y^3 \mapsto 0$, but

$$g(t^3, t^2)t^3 - h(t^3, t^2)t^2 \neq 0$$

because the t^3 term (which exists, because g has a constant term) cannot cancel. Therefore $x^2 - y^3 \nmid gx - hy$ and f is not algebraic. □

Let $V, W \subset \mathbb{C}^n$ affine algebraic varieties. Let $U \subset V \subset \mathbb{C}^n, \hat{U} \subset W \subset \mathbb{C}^m$ be Zariski-open subsets of the affine algebraic varieties.

Definition 2.5 $\Phi : U \to \hat{U}$ is algebraic (regular) if

1. Φ is continuous (with regard to the Z-top.)
2. $\forall S \underset{\text{open}}{\subset} \hat{U}$ and $\forall f : S \to \mathbb{C}$ algebraic the composition

$$\Phi^{-1}(S) \xrightarrow{\Phi} S \xrightarrow{f} \mathbb{C}$$

is algebraic.

Let Φ be an algebraic map. Then there is a correspondence

$$\frac{\mathbb{C}[y_1, \ldots, y_n]}{\mathcal{I}(W)} = \Gamma(W) \to \Gamma(V) = \frac{\mathbb{C}[x_1, \ldots, x_n]}{\mathcal{I}(V)}$$
$$f \mapsto f \circ \Phi$$

3 Projective varieties and morphisms

Definition 3.1 We define projective n-space over $\mathbb{C}$ as $\mathbb{P}^n := (\mathbb{C}^{n+1} \setminus \{0\})/\sim$, where the equivalence relation $\sim$ is defined by $x \sim y$ iff there is $\lambda \in \mathbb{C}^*$ with $y = \lambda x$.

Remark 3.2

1. $\mathbb{P}^n$ is not a vector space; it is the set of one-dimensional subspace of $\mathbb{C}^{n+1}$ (lines through the origin).
2. If $x \in \mathbb{P}^n$ is represented by $(x_0, \dots, x_n) \in \mathbb{C}^{n+1}$, then $(x_0, \dots, x_n)$ is a set of *homogenous coordinates* for x and denoted $[x_0, \dots, x_n]$ (unique only up to multiplication by a scalar), for any $\lambda \in \mathbb{C}^*$ we have $[\lambda x_0, \dots, \lambda x_n] = [x_0, \dots, x_n]$.

It is evident that polynomials $f \in \mathbb{C}[x_0, \dots, x_n]$ do not induce well-defined functions on complex projective n-space; there are however certain polynomials which allow meaningful discussion of their zero sets, also called *projective algebraic varieties*.

In addition, we can define the Zariski topology for projective varieties. The closed sets of this topology are defined to be the projective algebraic varieties.

First, we need the notion of an homogeneous ideal of the ring $\mathbb{C}[z_0, \dots, z_n]$. The motivation for this lies in the fact that our objects can only be described by homogeneous polynomials, and homogeneous ideals allow us to decompose any element into its homogeneous parts:

Definition 3.3 An ideal $I \subset \mathbb{C}[z_0, \dots, z_n]$ is homogeneous if for any $f \in I$ all the homogeneous components of f are again elements of I.

Example 3.4 Consider $f = f_0 + f_1 + \dots$ where f_i are the homogeneous components of degree i. Take for example: $f = 9 + 13z_0 + 3z_1^2$. In particular, f has a set of homogeneous generators.

We have the following results concerning the homogeneous ideal:

Lemma 3.5 Let $I \subset \mathbb{C}[z_0, \dots, z_n]$ be homogeneous, then I has finitely many homogeneous generators.

Proof. Let us consider $I \subset \mathbb{C}[z_0, z_1, z_2, \dots, z_n]$. Using Hilbert's basis theorem, we have that I is finitely generated in $\mathbb{C}[z_0, z_1, z_2, ...z_n]$ (say $I = (g_1, g_2, g_3, \dots, g_n)$). Taking the homogeneous components of the generators we obtain the conclusion ($I = (g_{00}, g_{01}, \dots, g_{10}, g_{11}, \dots)$). $\square$

Lemma 3.6 Consider an ideal $I \subset \mathbb{C}[z_0, \dots, z_n]$, which is generated by homogeneous elements. Then I is a homogeneous ideal.

Proof. Let $(F_1, F_2, F_3, \dots)$ be the homogeneous generators of I. It follows that $f \in I$ can be expressed $f = \sum_{i=1}^n g_i F_i$, where F_i are homogeneous polynomials of degree d_i and g_i are some polynomials (not necessarily homogeneous). The degree d part of f is it equal to $\sum_i G_i F_i$ where G_i represents the degree $d - d_i$ part of g_i. $\square$

Lemma 3.7 If I is a homogeneous ideal $I \subset \mathbb{C}[z_0, \dots, z_n]$ then $\sqrt{I}$ is homogeneous.

Proof. Consider $f \in \sqrt{I}$, then $\exists k \in \mathbb{N}$ such that $f^k \in I$. It follows that $f = f_l + f_d$ (where f_d is the maximal degree term of the polynomial, so $f^k = (f_l + f_d)^k$ can be written as $f^k = f_d^k +$ other terms. Then, $f^k - f_d^k \in \sqrt{I}$ and we can conclude (by induction). $\square$

Given the previous results we can consider $W \subset \mathbb{P}^n$ be an projective algebraic variety: the zero-locus of the homogeneous polynomials $F_1, F_2, F_3, \ldots, F_l$. Then it follows that the ideal $I = (F_1, F_2, F_3, \ldots, F_l)$ is a homogeneous ideal.

Theorem 3.8 (Projective Nullstellensatz) Let I be an ideal in $\mathbb{C}[z_0, \ldots, z_n]$ such that its zero locus $V(I)$ in $\mathbb{P}^n$ is non-empty. Then $\mathcal{I}(V(I)) = \sqrt{I}$.

Proof. As in the affine Nullstellensatz only $\mathcal{I}(V(I)) \subset \sqrt{I}$ requires proof; thus let $f \in \mathcal{I}(V(I))$. If $V(I) \subset \mathbb{P}^n$ is defined by the homogeneous polynomials $F_1, F_2, \ldots, F_k$, then these polynomials define an associated affine variety $\hat{V}(I)$ in $\mathbb{C}^{n+1}$. (As it is defined by homogenous polynomials, $\hat{V}(I)$ is a cone in the sense that $x \in \hat{V}(I)$ implies $\lambda x \in \hat{V}(I)$ for all $\lambda \in \mathbb{C}^*$.) It is clear that $f \in \mathcal{I}(V(I))$ vanishes on $\hat{V}(I) \setminus \{0\}$, for if $x \in \hat{V}(I) \setminus \{0\}$ then $[x] \in V(I)$. Now if $0 \in \hat{V}(I)$ (which always holds, unless one of the F_i is constant and non-zero) we use that there is $[x] \in V(I)$, giving $f(0) = 0$. Thus f vanishes on the whole of $\hat{V}(I)$; hence by the affine Nullstellensatz $f \in \mathcal{I}(\hat{V}(I)) = \sqrt{I}$. $\square$

Remark 3.9 The condition $V(I) \neq \emptyset$ is necessary, as can be seen by considering the maximal ideal $I := (z_0, \ldots, z_n)$ with $\hat{V}(I) = \{0\}$ in $\mathbb{C}^{n+1}$ and hence $V(I) = \emptyset$ in $\mathbb{P}^n$. We have $\mathcal{I}(V(I)) = (1)$, which is distinct from $\sqrt{I} = I$.

We will now introduce the notion of algebraic functions and maps. Note that they are also known as regular functions and maps, respectively.

Let $W \subset \mathbb{P}^n$ be a projective algebraic variety and U an open subset of W.

Definition 3.10 A function $f : U \to \mathbb{C}$ is called an algebraic function if for every $p \in U$ there is a neighborhood W of p in U and homogeneous polynomials of the same degree $h, g \in \mathbb{C}[z_0, \ldots, z_n]$ with $g \neq 0$ on W and $f|_W = h/g$.

Secondly, algebraic maps are defined as follows:

Definition 3.11

1. Suppose U is an open subset of either an affine algebraic variety or a projective algebraic variety. In this case, we call U a *quasi-projective* variety.
2. Let U, W be quasi-projective varieties. A map $f : U \to W$ is algebraic if:
   i) f is continuous w.r.t. the Zariski topology.
   ii) f respects precomposition with algebraic functions, i.e. $\forall Y \subset W$ open, $\forall g : Y \to \mathbb{C}$ algebraic : $g \circ f : f^{-1}(Y) \to \mathbb{C}$ is an algebraic function.

One easily checks that any open or closed subset of a quasi-projective variety is again a quasi-projective variety.

Remark 3.12 The reader should note the following basic facts about algebraic maps. The proofs are straightforward from the definitions and left as exercises.

1. If V is a quasi-projective variety, $f_1, \dots, f_n \in \Gamma(V)$ elements of the ring of algebraic functions on V, then

$$\begin{aligned} V &\to \mathbb{C}^n \text{ (or more generally: any affine algebraic variety)} \\ \xi &\mapsto (f_1(\xi), \dots, f_n(\xi)) \end{aligned}$$

   is an algebraic morphism.

2. If $V \subset \mathbb{P}^n$ is a quasi-projective variety, and if $F_0, \dots, F_m \in \mathbb{C}[Z_0, ..., Z_n]$ are homogeneous polynomials of the same degree, which moreover do not have a common zero on V, then

$$\begin{aligned} V &\to \mathbb{P}^m \\ \xi &\mapsto (F_0(\xi), \dots, F_m(\xi)) \end{aligned}$$

   defines an algebraic morphism.

3. If $\Phi : V \to W$ is an algebraic map between quasi-projective varieties, $\tilde{V} \subset V$ a subvariety, then the restriction

$$\Phi|_{\tilde{V}} : \tilde{V} \to W$$

   is algebraic. If $\tilde{W} \subset W$ is a subvariety that contains the image of Φ, then the map

$$\Phi : V \to \tilde{W}$$

   obtained by restricting the range of Φ is algebraic.

3.1 Morphisms of affine algebraic varieties

The study of the morphisms between affine algebraic varieties can be done by understanding their ring of functions. Given two affine algebraic varieties $V \subset \mathbb{C}^n$ and $W \subset \mathbb{C}^m$ with their corresponding ring of functions $\Gamma(V)$ and $\Gamma(W)$ we have the following correspondence:

If $\Phi : V \to W$ is an algebraic morphism, then there exists a homomorphism of $\mathbb{C}$-algebras (i.e. takes constants into constants):

$$\begin{aligned} \Phi^* : \Gamma(W) &\to \Gamma(V) \\ \gamma &\mapsto \gamma \circ \Phi \end{aligned}$$

Conversely, if there is a homomorphism of the $\mathbb{C}$-algebras $f : \Gamma(W) \to \Gamma(V)$, then this gives a unique algebraic morphism $\Phi : V \to W$ such that $\Phi^* = f$. Let

$$\begin{aligned} \Phi : V &\to \mathbb{C}^m \\ \xi &\mapsto (f(y_1)(\xi), \dots, f(y_m)(\xi)) \end{aligned}$$

where $y_1, \dots, y_m$ are the coordinate functions on W. We need to show that $\mathrm{Im}(\Phi) \subset W$. Let $g \in \mathcal{I}(W) \subset \mathbb{C}[Y_1, \dots, Y_m]$. Then

$$g \circ \Phi = g(f(y_1), \dots, f(y_m)) \underbrace{=}_{f \text{ homomorphism}} f(\underbrace{g(y_1, \dots, y_m)}_{=0}) = f(0) = 0$$

This implies $\mathrm{Im}(\Phi) \subset W$. Φ is an algebraic morphism by remark 3.12 1. (see also Lemma 3.6 in Hartshorne). Let $\gamma \in \Gamma(W)$. Then

$$\Phi^*(\gamma)(\xi) = \gamma(f(y_1)(\xi), \dots, f(y_m)(\xi)) \underbrace{=}_{f \text{ homomorphism}} f(\gamma(y_1, \dots, y_m))(\xi)$$

and we have $\Phi^* = f$.
In conclusion, Φ and Φ^* encode the same data.

Example 3.13 • Let

$$\begin{aligned}\Phi : \mathbb{C} &\to \mathbb{C}^2 \\ t &\mapsto (t^2, t^3)\end{aligned}$$

Then $\mathrm{Im}(\Phi) = V(x^3 - y^2)$. This is an affine algebraic variety.

- Let

$$\begin{aligned}\Phi : \mathbb{C}^2 &\to \mathbb{C}^2 \\ (x, y) &\mapsto (x, xy)\end{aligned}$$

Then

$$\mathrm{Im}(\Phi)^C = \{(0, \rho) | \phi \neq 0\}$$

and $\mathrm{Im}(\Phi)$ is not a quasi-projective variety.

It follows that the image of an algebraic map need not be a quasi-projective variety.

4 Projective varieties and morphisms II

Recall that we defined a quasi-projective variety as an open subset of either a projective or an affine variety. This makes sense, as we will see that any open subset of an affine variety is isomorphic to, i.e. can be viewed as an open subset of a projective variety.

Let $n \geq 1$. For $0 \leq i \leq n$, we have basic open subsets

$$U_i := U_{Z_i} = \{ \ [\xi_0, \xi_1, \ldots, \xi_n] \in \mathbb{P}^n \mid \xi_i \neq 0\}$$

of projective space, and maps Φ_i,Ψ_i defined by:

$$\begin{aligned}\Phi_i : \mathbb{C}^n &\to U_i \\ (x_1, \ldots, x_n) &\mapsto [x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n]\end{aligned}$$

and

$$\begin{aligned}\Psi_i : U_i &\to \mathbb{C}^n \\ [\xi_0, \xi_1, \ldots, \xi_n] &\mapsto \left(\frac{\xi_0}{\xi_i}, \ldots, \frac{\xi_{i-1}}{\xi_i}, \frac{\xi_{i+1}}{\xi_i}, \ldots, \frac{\xi_n}{\xi_i}\right)\end{aligned}$$

which is well defined on U_i. Note that the U_i cover all of projective space.

Claim 4.1 Φ_i and Ψ_i are isomorphisms between varieties, which are inverse to each other.

Proof. It is easy to check that the composition of Ψ_i with Φ_i in any order is the identity map. So one only needs to show that both are algebraic maps. For the sake simplicity, we consider only the case when i is zero and drop the index 0 when referring to Ψ_0, Φ_0.

1. To prove continuity, we verify that the preimages of closed sets remain closed under both maps.

   Let first $W \subset U_0$ be a closed subset defined by homogeneous polynomials $F_1, \ldots, F_k \in \mathbb{C}[Z_0, \ldots, Z_n]$. Then $\Phi^{-1}(W)$ is the common zero set of the set of polynomials

   $$(F_i(1, X_1, \ldots, X_n))_{i=1,\ldots,k} \in \mathbb{C}[X_0, \ldots, X_n]$$

   (check this!), thus closed in $\mathbb{C}^n$.

   Conversely, if $V \subset \mathbb{C}^n$ is defined by $(f_i)_{i=1,\ldots,k} \in \mathbb{C}[X_1, \ldots, X_n]$, then we can construct homogeneous polynomials $(F_i)_{i=1,\ldots,k}$ in $n+1$ variables, by $F_i[Z_0, ..., Z_n] := f_i(\frac{Z_1}{Z_0}, \ldots, \frac{Z_n}{Z_0}) Z_0^{\deg f}$. Check that these define the preimage $\Psi^{-1}(V)$ in U_0 as their set of common zeros.

2. Let $f : U \to \mathbb{C}$ be any function, where U is open in U_0. Let $W := \Phi^{-1}(U)$. We want to show that $f \circ \Phi : W \to \mathbb{C}$ is regular if and only if f is. Note that $f \circ \Phi(x_1, \ldots, x_n) = f([1, x_1, \ldots, x_n])$ and $f([\xi_0, \ldots, \xi_n]) = f \circ \Phi(\frac{\xi_1}{\xi_0}, \ldots, \frac{\xi_n}{\xi_0})$.

   Assume f is regular. Let $x = (x_1, \ldots, x_n) \in W$, $\xi := \Phi(x)$. By assumption, there is an open neighborhood $\tilde{U}$ of ξ, such that $f|_{\tilde{U}} = \frac{H}{G}|_{\tilde{U}}$, where H and $G \in \mathbb{C}[Z_0, \ldots, Z_n]$ are homogeneous polynomials of the same degree, and G doesn't vanish on $\tilde{U}$. Then $f \circ \Phi$ equals $\frac{H(1,X_1,\ldots,X_n)}{G(1,X_1,\ldots,X_n)}$ on the preimage of $\tilde{U}$, which is an open neighborhood of x. Noting that $G(1, X_1, \ldots, X_n)$ is never zero for any point in $\Phi^{-1}(\tilde{U})$, this means that we have found a neighborhood of x where $f \circ \Phi$ is equal to a rational function. This proves that Φ is algebraic.

   The other direction works similarly: Assume $f \circ \Psi^{-1}$ is algebraic. Let $\xi \in U, x = \Psi(\xi)$. By assumption, there is an open neighborhood $\tilde{W}$ of x in W on which $f \circ \Psi^{-1}$ equals a rational function $\frac{h}{g}$, and g does not vanish on $\tilde{W}$. Now $\tilde{U} := \Phi(\tilde{W}) = \Psi^{-1}(\tilde{W})$ is an open neighborhood of ξ in W, and we have by the above remark:

$$f|_{\tilde{U}} = \frac{h(\frac{Z_1}{Z_0}, \ldots, \frac{Z_n}{Z_0})}{g(\frac{Z_1}{Z_0}, \ldots, \frac{Z_n}{Z_0})} = \frac{h(\frac{Z_1}{Z_0}, \ldots, \frac{Z_n}{Z_0})Z_0^D}{g(\frac{Z_1}{Z_0}, \ldots, \frac{Z_n}{Z_0})Z_0^D},$$

   which holds for any positive integer D. In particular if $D \geq \max\{\deg h, \deg g\}$, then both the denominator and the numerator are homogeneous polynomials of the same degree in $Z_0, Z_1, \ldots, Z_n$. As ξ was arbitrary, this proves that Ψ is algebraic.

□

From what we have proven we directly obtain

Lemma 4.2 Any projective variety can be covered by finitely many affine varieties.

Using our newfound knowledge, we now want to compute the algebraic functions on $\mathbb{CP}^n$.

Lemma 4.3 Any algebraic function $f : \mathbb{CP}^n \to \mathbb{C}$ is constant.

Proof. Let $f : \mathbb{CP}^n \mapsto \mathbb{C}$ be algebraic. For $1 \leq i \leq 1$, the restriction of f to U_i gives an algebraic function

$$\mathbb{C}^n \to \mathbb{C} \tag{1}$$

$$(x_1, \ldots, x_n) \mapsto f(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n), \tag{2}$$

but we already know that such a function must equal a polynomial $P_i \in \mathbb{C}[X_1, \ldots, X_n]$.

Let

$$Q_i(Z_0, \ldots, Z_n) := Z_i^d P_i\left(\frac{Z_0}{Z_i}, \ldots, \frac{Z_{i-1}}{Z_i}, \frac{Z_{i+1}}{Z_i}, \ldots, \frac{Z_n}{Z_i}\right),$$

where d is a positive integer that is strictly bigger than the maximal degree of P_i. Q_i is homogeneous of degree d and vanishes outside U_i. By construction,

$$f|_{U_i} = \frac{Q_i(Z_0, Z_1, \ldots, Z_n)}{Z_i^d},$$

and in particular on the overlap regions:

$$f|_{U_0 \cap U_i} = \frac{Q_0(Z_0, \ldots, Z_n)}{Z_0^d} = \frac{Q_i(Z_0, \ldots, Z_n)}{Z_i^d}.$$

So the equation

$$Z_i^d Q_0(Z_0, \ldots, Z_n) = Z_0^d Q_i(Z_0, \ldots, Z_n)$$

is fulfilled at every point in $\mathbb{C}^{n+1}$, where the zero'th and i'th coordinate are not equal to zero. However, at all other points, we see that both sides of the equations are zero. This means, the two polynomials take the same values on all of $\mathbb{C}^n$, which implies they are the same. By unique factorization we get, for $0 \leq i \leq n$, that Z_i^d divides Q_i, and both have degree d, which means the quotient

$$\frac{Q_i}{Z_i^d} =: c_i,$$

has to be a constant, and by the above equality, $c_i = c_0$ for any i. As the U_i cover $\mathbb{CP}^n$, f is everywhere constant and equal to c_0.

□

Corollary 4.4 $\mathbb{CP}^n$ is not isomorphic to any affine algebraic variety.

Proof. An isomorphism between varieties induces a isomorphism of $\mathbb{C}$-algebras between their rings of functions. The only affine varieties whose ring of functions equals $\mathbb{C}$ are the points. But $\mathbb{CP}^n$ is not isomorphic to a point. □

5 Veronese embedding

The discussion so far has been about quasi-projective varieties and their regular functions $f : X \to \mathbb{C}$ as well as their regular maps $p : X \to Y$. We are interested to know what the image of a regular map of a projective variety is.

Suppose we have a map $\mathbb{P}^n \to A$, where A is an affine variety. What are the possibilities for this map? The claim is that this map must be constant. To see this, assume that for two points $\xi_1 \neq \xi_2 \in \mathbb{P}^n$ we have $f(\xi_1) \neq f(\xi_2)$. This means $f(\xi_1)$ and $f(\xi_2)$ differ in at least one coordinate x_i. Define a map $g : A \to \mathbb{C}$ by $g(x) = x_i$. Then the composition $gf : \mathbb{P} \to \mathbb{C}$ consists of a regular non-constant map, which we have already proved does not exist.

Now consider regular maps $\mathbb{P}^n \to \mathbb{P}^m$. They are given by $[z_0, \ldots, z_n] \to [F_0, \ldots, F_m]$ where all F_i are homogeneous polynomials in $\mathbb{C}[z_0, \ldots, z_n]$ of the same degree and moreover have no common zeros.

Definition 5.1 The *Veronese* embedding of degree $d \in \mathbb{N}$ is the map

$$\begin{aligned} \nu_d : \mathbb{P}^n &\to \mathbb{P}^{\binom{d+n}{n}-1} \\ [z_0, \ldots, z_n] &\mapsto [(z_0^{i_0} \cdots z_n^{i_n})_{0 \leq i_k \leq d, \sum_{k=0}^n i_k = d}]. \end{aligned}$$

Example 5.2 The map

$$\begin{aligned} \mathbb{P}^1 &\to \mathbb{P}^2 \\ [x, y] &\mapsto [x^2, xy, y^2] \end{aligned}$$

is the Veronese of degree 2 mapping from $\mathbb{P}^1$.

Example 5.3 The Veronese with $n = 1, d = 3$

$$\begin{aligned} \mathbb{P}^1 &\to \mathbb{P}^3 \\ [x, y] &\mapsto [x^3, x^2y, xy^2, y^3] \end{aligned}$$

is also called *twisted cubic*.

Let $X_{i_0,\ldots,x_n}$ be the variable in $\mathbb{P}^{\binom{n+d}{n}-1}$ that corresponds to the monomial $z_0^{i_0} \cdots z_n^{i_n}$ in the Veronese map. Then, we have on the image of the Veronese map, for the multi-indices $I = i_0, \ldots, i_n, J = j_0, \ldots, j_n, K = k_0, \ldots, k_n, L = l_0, \ldots, l_n$ such that $I + J = L + K$:

$$X_I \cdot X_J - X_K \cdot X_L = 0 \tag{3}$$

Proposition 5.4 The image of the Veronese is defined by the quadratic equations (3).

Corollary 5.5 The image of each Veronese map is a projective algebraic variety.

5.1 Linear maps and Linear Hypersurfaces in $\mathbb{P}^n$

Definition 5.6 A *linear hypersurface* in $\mathbb{P}^n$ is defined by one equation:

$$0 = \sum_{k=0}^{n} a_k z_k.$$

Remark 5.7 When we deal with one hypersurface, we can assume $a_0 = 0$, by a change of coordinates.

Definition 5.8 The *projective linear group* is $\mathrm{PGL}_{n+1}(\mathbb{C}) = \mathrm{GL}_{n+1}(\mathbb{C})/\mathbb{C}^*$.

Remark 5.9 $\mathrm{PGL}_{n+1}(\mathbb{C})$ is the group of automorphisms of $\mathbb{P}^n$.

Because homogeneous coordinates are unique only up to a factor $\lambda \in \mathbb{C}^*$, matrices defining a linear map $\mathbb{P}^n \to \mathbb{P}^n$ also define the same map if they differ only by some factor $\lambda \in \mathbb{C}^*$.

5.2 Quadratic hypersurfaces

Definition 5.10 A *quadratic hypersurface* or *quadric* in projective space $\mathbb{P}^n$ is the zero locus of some $F \in \mathbb{C}[z_0, \ldots, z_n]$ of the form

$$F = \sum_{k=0}^{n} a_k z_k^2 + \sum_{k=0}^{n} \sum_{l=k+1}^{n} a_{k,l} z_k z_l.$$

Remark 5.11 A quadric

$$F = \sum_{k=0}^{n} a_k z_k^2 + \sum_{k=0}^{n} \sum_{l=k+1}^{n} 2a_{k,l} z_k z_l$$

can also be written using a symmetric matrix M:

$$F = \begin{pmatrix} z_0 & \cdots & z_n \end{pmatrix} \underbrace{\begin{pmatrix} a_0 & a_{0,1} & \cdots & a_{0,n} \\ a_{0,1} & a_1 & \cdots & a_{1,n} \\ \vdots & & \ddots & \vdots \\ a_{0,n} & \cdots & a_{n-1,n} & a_n \end{pmatrix}}_{=:M} \begin{pmatrix} z_0 \\ \vdots \\ z_n \end{pmatrix}$$

Definition 5.12 The *rank* of a quadric is the rank of the symmetric matrix M associated with it.

Proposition 5.13 A quadric is determined, up to $\mathrm{PGL}_{n+1}(\mathbb{C})$ by its rank.

Proof. M be the symmetric matrix of a quadric. Then, there is a $P \in \mathrm{PGL}_{n+1}(\mathbb{C})$ such that

$$(P^T)^{-1} M P^{-1} = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}$$

where the number of 1's is equal to the rank of M. Let $Q = \{x | x^T M x = 0\}$ be the quadric associated with M. Then

$$\begin{aligned} PQ &= \{Px | x^T M x = 0\} \\ &= \{x | x^T (P^T)^{-1} M P^{-1} x = 0\} \qquad = \left\{ x \,\middle|\, x^T \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix} x = 0 \right\} \end{aligned}$$

□

Something about the genus of cubic and quadratic surfaces is missing here.

6 Elliptic functions and cubic curves

There is an intimate connection between cubic equations and elliptic functions (doubly periodic meromorphic functions), in particular the Weierstrass $\wp$-function. Consider a meromorphic function f on $\mathbb{C}$ which is doubly-periodic with periods 1 and i; let Λ denote the lattice in $\mathbb{C}$ generated by 1 and i (which is, as a set, simply $\mathbb{Z}[i]$), defining also $\Lambda' := \Lambda - \{0\}$. (One could consider instead of i any element of the upper half plane.) It follows that we also have

$f(z) = f(z+\lambda)$ for all $\lambda \in \Lambda$. It is evident that f is determined by its values on the *fundamental parallelogram* $\Gamma := [0,1) \times [0,i)$. If f is an entire function, then f is bounded on the compact set $\overline{\Gamma}$ by continuity, hence on all of $\mathbb{C}$ so that f must be constant by Liouville's theorem. If f has only a simple pole ξ in the interior of Γ, then the residue theorem gives (taking $\partial\Gamma$ to be positively oriented)

$$\frac{1}{2\pi i}\int_{\partial\Gamma} f(z)dz = \mathrm{res}_\xi f.$$

The integral vanishes as opposite sides of $\partial\Gamma$ cancel one another by periodicity; hence $\mathrm{res}_\xi f = 0$ meaning that ξ is a removable singularity. Thus the argument above applies and shows f to be constant. Therefore, if f is to be nonconstant it must have at least two poles in Γ.

We now define the elliptic function $\wp$, which has double poles at the points of Λ. It is defined by the infinite series

$$\wp(z) := \frac{1}{z^2} + \sum_{\omega\in\Lambda'}\left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right),$$

which converges compactly for all $z \notin \Lambda$ and hence defines a meromorphic function on $\mathbb{C}$. As we may differentiate termwise, the derivative is

$$\wp'(z) = -\frac{2}{z^3} - \sum_{\omega\in\Lambda'}\frac{2}{(z-\omega)^3} = -2\sum_{\omega\in\Lambda}\frac{1}{(z-\omega)^3}.$$

As the sum is invariant under $z \mapsto z+\lambda$ ($\lambda \in \Lambda$), we have $\wp'(z) = \wp'(z+\lambda)$ for all $\lambda \in \Lambda$, hence $\wp'$ is an elliptic function. From this it follows that $\wp(z) - \wp(z+\lambda)$ is constant and in fact vanishes identically as $\wp(-\lambda/2) - \wp(\lambda/2) = 0$ ($\lambda/2 \notin \Lambda$ is not a pole), since $\wp$ is even. Thus $\wp$ is indeed doubly-periodic with periods 1 and i.

If f is an elliptic function with periods 1 and i (not vanishing on $\partial\Gamma$ and without poles on $\partial\Gamma$), then the integral

$$\frac{1}{2\pi i}\int_{\partial\Gamma}\frac{f'(z)}{f(z)}dz$$

vanishes by an argument as in the first paragraph. By the argument principle this integral is equal to the number of zeros of f minus the number of poles of f (both counted with multiplicity). Thus $\wp$ has equally many poles as it has zeros.

The Laurent expansion of $\wp$ about the origin is

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega\in\Lambda'}\left(\frac{1}{\omega^2}\left(\sum_{k=0}^{\infty}\frac{z^k}{w^k}\right)^2 - \frac{1}{\omega^2}\right) = \frac{1}{z^2} + \sum_{n=1}^{\infty} c_n z^n \quad \text{where} \quad c_n = (n+1)\sum_{\omega\in\Lambda'}\frac{1}{\omega^{n+2}}$$

vanishes for n odd. Thus this can also be written as

$$\wp(z) = \frac{1}{z^2} + \sum_{k=2}^{\infty}(2k-1)G_k z^{2k-2} \quad \text{where} \quad G_k := \sum_{\omega\in\Lambda'}\frac{1}{\omega^{2k}},$$

is the Eisenstein series of weight $2k$ (at i), and we find

$$\begin{aligned}\wp'(z)^2 &= \frac{4}{z^6} - \frac{24G_2}{z^2} - 80G_3 + \dots\\ 4\wp^3 &= \frac{4}{z^6} + \frac{36G_2}{z^2} + 60G_3 + \dots;\end{aligned}$$

defining $g_2 := 60G_2$ and $g_3 := 140G_3$ we see that the differential equation

$$\wp'^2 = 4\wp^3 - g_2\wp - g_3$$

holds, since $\wp'(z)^2 - 4\wp(z)^3 + 60G_2\wp(z) = -140G_3 + \ldots$, the right hand side being an elliptic function without poles (and is hence constant).

Thus the points of the form $(\wp(z), \wp'(z))$ satisfy the cubic equation $y^2 = 4x^3 - g_2x - g_3$. In fact if we define $\phi : \mathbb{C} \to \mathbb{P}^2$ by $\phi(z) = [1, \wp(z), \wp'(z)]$ we get an induced bijection between the cubic curve defined by $y^2 = 4x^3 - g_2x - g_3$ (embedded into $\mathbb{P}^2$) and the commutative group $\mathbb{C}/\Lambda$. The latter is homeomorphic to a torus, as can be seen by considering the construction of the torus from its fundamental polygon; notice that the geometric structure of a torus is intrinsically doubly periodic. The cubic curve is in fact nonsingular (the roots of the polynomial $4x^3 - g_2x - g_3$ are distinct are in fact given by $\wp(1/2)$, $\wp(i/2)$, $\wp(1/2 + i/2)$), hence an elliptic curve and as such has a group law (turning the bijection into a group isomorphism).

Ahlfors' complex analysis, chap. 7, was recommended during the lecture. It was also suggested to read the entire book.

7 Intersections of lines with curves

Definition 7.1 Let $\alpha = (\alpha_0, \ldots, \alpha_n)$, $\beta = (\beta_0, \ldots, \beta_n) \in \mathbb{C}^n$ be linearly independent vectors. A set $S \subset \mathbb{P}^n$ of the form

$$S := \{[\alpha_0 s + \beta_0 t, \alpha_1 s + \beta_1 t, \ldots, \alpha_n s + \beta_n t] | [s, t] \in \mathbb{P}^1\}$$

is called a *line*.

Let $F \in \mathbb{C}[X, Y, Z]$ be homogeneous of degree d. We want to find the intersection of the zeros of F with the a line given by $(\alpha_0, \beta_0, \gamma_0), (\alpha_1, \beta_1, \gamma_1)$. This is given by

$$\underbrace{F(\alpha_0 s + \alpha_1 t, \beta_0 s + \beta_1 t, \gamma_0 s + \gamma_1 t)}_{=\hat{F}(s,t)} = 0$$

Either $\hat{F} = 0$ or

$$\hat{F} = (\lambda_0^1 s + \lambda_1^1 t)(\lambda_0^2 s + \lambda_1^2 t) \cdots (\lambda_0^d s + \lambda_1^d t)$$

8 Products of varieties and the Segre embedding

We would like to investigate the products of algebraic varieties. In the affine case, this is a very simple matter. For $f_1, \ldots, f_r \in \mathbb{C}[x_1, \ldots, x_n]$, $g_1, \ldots, g_s \in \mathbb{C}[y_1, \ldots, y_m]$, let $A = V(\langle f_1, \ldots, f_r \rangle) \subset \mathbb{C}^n$ and $B = V(\langle g_1, \ldots, g_s \rangle) \subset \mathbb{C}^m$ be affine varieties. Then $A \times B$ is a subset of $\mathbb{C}^n \times \mathbb{C}^m$, which in turn is isomorphic to $\mathbb{C}^{n+m}$. We can now consider the f_i and g_i as elements of $\mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_m]$. It is clear that $A \times B = V(\langle f_1, \ldots, f_r, g_1, \ldots, g_s \rangle)$.

In the projective case, the above approach does not work, as $\mathbb{P}^n \times \mathbb{P}^m$ is not isomorphic to $\mathbb{P}^{n+m}$. However, we can define an embedding from $\mathbb{P}^n \times \mathbb{P}^m$ to a projective space.

Definition 8.1 The map

$$\sigma : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{(n+1)(m+1)-1}$$
$$[x_0, x_1, \ldots, x_n] \times [y_0, y_1, \ldots, y_m] \mapsto [x_0y_0, x_0y_1, \ldots, x_iy_j, \ldots x_ny_m],$$

where $0 \leq i \leq n$ and $0 \leq j \leq m$, is called the *Segre embedding*.
We give $\mathbb{P}^n \times \mathbb{P}^m$ the structure of a projective variety by identifying it with its image.

We will prove that our definitions make sense:

Claim 8.2 σ is injective.

Proof. Let $c = [c_{0,0}, c_{0,1}, \ldots, c_{i,j}, \ldots c_{n,m}]$ be an element of $\mathrm{Im}(\sigma)$. Let $(a, b) \in \mathbb{P}^n \times \mathbb{P}^m$ such that $\sigma(a, b) = c$. Without loss of generality, $a_0 = b_0 = c_{0,0} = 1$. This enforces $b_j = c_{0,j}$ for all $0 \leq j \leq m$ and $a_i = c_{i,0}$, which uniquely determines a and b. Therefore, the Segre embedding is bijective onto its image. □

Claim 8.3 The image of σ is a variety of $\mathbb{P}^{(n+1)(m+1)-1}$.

Proof. We denote the complex polynomials in $(n+1)(m+1)$ variables by $\mathbb{C}[z_{0,0}, z_{0,1}, \ldots, z_{i,j}, \ldots z_{n,m}]$. For $i \neq k$ and $j \neq l$, let us define $P_{i,j,k,l} := z_{i,j}z_{k,l} - z_{k,j}z_{i,l}$ and I, the ideal generated by the $P_{i,j,k,l}$. It is clear that $\sigma(\mathbb{P}^n \times \mathbb{P}^m) \subset V(I)$. For the other inclusion, take a $c \in V(I)$. Without loss of generality, $c_{0,0} = 1$. For any $k, l \neq 0$, the polynomial $P_{0,0,k,l}$ gives us $c_{k,l} = c_{k,0}c_{0,l}$. By taking $a_0 = b_0 = 1$, $a_k = c_{k,0}$ and $b_l = c_{0,l}$, we get (a, b) such that $\sigma(a, b) = c$. □

Remark 8.4

1. By definition, the Segre embedding is algebraic. This, together with Claim 8.2, implies that the Segre embedding is an isomorphism onto its image.
2. It is also possible to define an algebraic structure on $\mathbb{P}^n \times \mathbb{P}^m$ by using bihomogeneous polynomials. One can check that this is equivalent to our definition.
3. We can also define the product of two quasi-projective varieties X and Y. It is not very hard to check that $\sigma(X \times Y)$ again is a quasi-projective variety.
4. If X and Y are quasi-projective, it is useful to know that their product $X \times Y$ is a categorical product, i.e. the projections on the first resp. second coordinate are algebraic maps, and if we have algebraic maps

$$\begin{aligned} f &: Z \to X \\ g &: Z \to Y \end{aligned}$$

then the product map

$$\begin{aligned} (f, g) : Z &\to X \times Y \\ z &\mapsto (f(z), g(z)) \end{aligned}$$

is also algebraic.

Example 8.5 Given four skew lines L_1, L_2, L_3, L_4 in $\mathbb{P}^3$, how many lines intersect all of them? Consider the vector space V of homogeneous polynomials of degree two in the variables x, y, z, w. This is a 10 dimensional space and its projectivization parametrizes quadrics in $\mathbb{P}^3$. Pick three generic points on each of the lines L_1, L_2, L_3. The condition for a quadric $Q \in \mathbb{P}(V)$ to contain a point is a linear condition on V. As the lines are chosen general and there are three points on each line, there will be a 1-dimensional subspace of V of homogeneous polynomials vanishing at all the points and therefore a unique quadric Q that contains the chosen points (Check that the conditions are independent!). The restriction of the polynomial that defines Q, to the lines L_i, $i = 1, 2, 3$ is a polynomial of degree 2 that vanishes at three distinct points, hence must be zero. This shows that Q contains the three lines L_1, L_2, L_3.

As the lines chosen are generic, Q is a smooth quadric and after a change of coordinates we may assume it is given by $V(xw - yz)$, which is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ via the Segre map. The lines L_1, L_2, L_3 are then given $\mathbb{P}^1 \times P_i$ for $P_i \in \mathbb{P}^1$, $i = 1, 2, 3$ some points (This follows as they are distinct and of degree 1). Any line that meets the three lines L_1, L_2, L_3 must meet the quadric Q in three distinct points and with the same argument as above is contained in Q. It must hence given by $M_R = R \times \mathbb{P}^1$ for $R \in \mathbb{P}^1$ a point.

Consider now the fourth general line L_4. L_4 meets the quadric Q in precisely 2 points $(R_1, S_1), (R_2, S_2) \in \mathbb{P}^1 \times \mathbb{P}^1$. This shows that there are precisely two lines (namely M_{R_1} and M_{R_2}) incident to all four lines $L_1, \ldots, L_4$.

8.1 Four lines in $\mathbb{P}^3$

Given four skew (i.e. pairwise non-intersecting) lines $L_1, \ldots, L_4$ in $\mathbb{P}^3$, how many lines intersect all of them?

Lemma 8.6 Let L_1, L_2, L_3 be skew lines in $\mathbb{P}^3$ and let K_1, K_2, K_3 be skew lines in $\mathbb{P}^3$. Then, there is a $M \in \mathrm{PGL}_4(\mathbb{C})$ such that $ML_i = K_i$ for $i = 1, 2, 3$.

Proof. The ring of polynomials of $\mathbb{P}^3$ is $\mathbb{C}[X, Y, Z, W]$. Without loss of generality we may assume that

$$K_1 = Z(X, Y), K_2 = Z(Z, W), K_3 = Z(X - Z, Y - W).$$

Because each line can be defined by two points that are on the line, we can find $M' \in \mathrm{PGL}_4(\mathbb{C})$ such that $M'L_1 = K_1, M'L_2 = K_2$, because M' can be defined by mapping four points in general linear position to four points in general linear position.
Now, consider $M'L_3$. This cannot lie in $X = 0$ or $W = 0$, because then $M'L_3$ and $M'L_2$ would not be skew or $M'L_3$ and $M'L_1$ would not be skew. Therefore $M'L_3$ intersects $X = 0$ and $W = 0$ in one point each. Let $[0 : a : b : c]$ be the point where $M'L_3$ intersects $X = 0$ and let $[d : e : f : 0]$ be the point where $M'L_3$ intersects $W = 0$. K_3 is defined by the points $[0 : 1 : 1 : 1]$ and $[1 : 1 : 1 : 0]$. Now, we need some $N \in \mathrm{PGL}_4(\mathbb{C})$ that leaves K_1, K_2 fixed and maps $[0 : a : b : c]$ to $[0 : 1 : 1 : 1]$ and $[d : e : f : 0]$ to $[1 : 1 : 1 : 0]$. N can be written as a block matrix

$$N = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where A, B, C, D are 2×2 matrices. To fix K_1 we need that $N[0 : 0 : z_2 : z_3] = [0 : 0 : z_2' : z_3']$. This implies $B = 0$. Fixing K_3, similarly, implies $C = 0$. To map $[0 : a : b : c]$ to $[0 : 1 : 1 : 1]$ and $[d : e : f : 0]$ to $[1 : 1 : 1 : 0]$ we can set

$$N = \begin{bmatrix} 1/d & 0 & 0 & 0 \\ 0 & 1/a & 0 & 0 \\ 0 & 0 & 1/f & 0 \\ 0 & 0 & 0 & 1/c \end{bmatrix}.$$

Set $M = NM'$. □

Lemma 8.7 Let L_1, L_2, L_3 be skew lines in $\mathbb{P}^3$. Then, the family of lines which meets all three lines sweeps out a quadric in $\mathbb{P}^3$.

Proof. By the previous lemma we can assume that

$$L_1 = Z(X, Y), L_2 = Z(Z, W), L_3 = Z(X - Z, Y - W)$$

Let $Q = V(XW - YZ)$. Then Q contains L_1, L_2, L_3. Also Q is the image of the Segre map from $\mathbb{P}^1 \times \mathbb{P}^1$. We have that

$$L_1 = \sigma(\{[0 : 1]\} \times \mathbb{P}^1), L_2 = \sigma(\{[1 : 0]\} \times \mathbb{P}^1), L_3 = \sigma(\{[1 : -1]\} \times \mathbb{P}^1)$$

For any $P \in \mathbb{P}^1$ the line $\sigma(\mathbb{P}^1 \times P)$ intersects L_1, L_2, L_3 and so

$$Q = \sigma(\mathbb{P}^1 \times \mathbb{P}^1) = \bigcup_{P \in \mathbb{P}^1} \sigma(\mathbb{P}^1 \times \{P\})$$

and therefore the lines which intersect all of L_1, L_2, L_3 sweep out at least Q. Let L_4 be a line that intersects all of L_1, L_2, L_3. Let $P_1 \in L_4 \cap L_1, P_2 \in L_4 \cap L_2, P_3 \in L_4 \cap L_3$. Then $XW - YZ$ is zero on P_1, P_2, P_3, and therefore it is zero on all of L_4 as it has degree 2. □

9 Intersections of quadrics

Let $Q_1, Q_2 \subset \mathbb{P}^3$ be quadrics. We want to determine $Q_1 \cap Q_2$. We use the Segre to cut out the first quadric:

$$\begin{aligned} S : \mathbb{P}^1 \times \mathbb{P}^1 &\to \mathbb{P}^3 \\ [x_0, x_1] \times [y_0, y_1] &\mapsto [x_0y_0, x_0y_1, x_1y_0, x_1y_1] =: [z_0, z_1, z_2, z_3] \end{aligned}$$

The image is the zeros of $z_0z_3 - z_1z_2$.

Step 1 Choose coordinates, such that

$$Q_1 \underbrace{\cong}_{\text{Segre}} \mathbb{P}^1 \times \mathbb{P}^1$$

Step 2 Now consider $S^{-1}(Q_2) \subset \mathbb{P}^1 \times \mathbb{P}^1$. $Q_2 \subset \mathbb{P}^3$ is defined by the equation

$$c_1z_0^2 + c_2z_1^2 + c_3z_2^2 + c_4z_3^3 + c_5z_0z_1 + \cdots + c_{10}z_2z_3 =: F(z_0, z_1, z_2, z_3)$$

What is $S^{-1}(Q_2)$? Let

$$G(x_0, x_1, y_0, y_1) = F(x_0y_0, x_0y_1, x_1y_0, x_1y_1)$$

Let $p \in \mathbb{P}^1 \times \mathbb{P}^1$. Then $p \in S^{-1}(Q_2) \Leftrightarrow G(p) = 0$. But being zero for G doesn't even make sense if it is (only) homogeneous. It needs to be bihomogeneous (i.e. homogeneous in each set of variables separately).

Definition 9.1 A polynomial $p(x_1, \ldots, x_n, y_1, \ldots, y_m)$ is bihomogeneous of bidegree (a, b) if it is homogeneous of degree a in the x's and homogeneous of degree b in the y's.

Example 9.2 $x_0x_1y_0^3$ is bihomogeneous with bidegree $(2, 3)$.

If G is bihomogeneous of bidegree (a, b), then whether $G(p) = 0$ is well defined for $p \in \mathbb{P}^1 \times \mathbb{P}^1$. Let

$$p = [x_0, x_1] \times [y_0, y_1] = [\lambda x_0, \lambda x_1] \times [\mu y_0, \mu y_1]$$

Then $G(p)$ is defined up to a factor of $\lambda^a \cdot \mu^b$.

Proposition 9.3 Let G be bihomogeneous. $Z(G) \subset \mathbb{P}^1 \times \mathbb{P}^1$ is an algebraic subvariety.

Proof. Trivial. □

We have that $Q_2 = F(z_0, z_1, z_2, z_3)$ is homogenenous of degree 2. Then we have $G(x_0, y_1, y_0, y_1) = F(x_0y_0, x_0y_1, x_1y_0, x_0y_1)$ is bihomogeneous of degree $(2, 2)$. We have projection maps $\pi_1, \pi_2 : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ for the x and the y coordinates respectively. Look at $\pi_2(Z(G)) \subset \mathbb{P}^1$. What are the fibers of $Z(G)$? For $[\xi_0, \xi_1] \in \mathbb{P}^1$, $G(x_0, x_1, \xi_0, \xi_1)$ has either one or two roots as a polynomial in $[x_0, x_1] \in \mathbb{P}^1$. Can $G(x_0, x_1, y_0, y_1)$ be zero for all $[x_0, x_1]$? We have

$$G(x_0, x_1, y_0, y_1) = x_0^2 P_1(\xi_0, \xi_1) + x_0x_1P_2(\xi_0, \xi_1) + x_1^2P_3(\xi_0, \xi_1)$$

where $P_1, P_2, P_3 \in \mathbb{C}[y_0, y_1]$, homogeneous of degree 2. This can be zero for all $[x_0, x_1]$ if and only if $[\xi_0, \xi_1]$ is a common root of P_1, P_2, P_3.

How many points of $\pi_2(Z(G))$ are double (ramification)? Look at

$$\begin{aligned} m_G : \mathbb{P}^1_y &\to \mathbb{P}^2 \\ [y_0, y_1] &\mapsto [P_1(y_0, y_1), P_2(y_0, y_1), P_3(y_0, y_1)] \end{aligned}$$

Consider the quadratic equation $P_1x_0^2 + P_2x_0x_1 + P_3x_1^2$ in $\mathbb{P}^2$. We have a double zero if the discriminant $P_2^2 - 4P_1P_3$ is zero. this has 4 solutions in general, because it is an equation of degree 4 in $\mathbb{P}^1$.
Something about the Euler characteristic is missing here!

10 The Grassmannian and the incidence correspondence

Quadrics in different dimensions

- In $\mathbb{P}^1$ we just have 2 points.
- In $\mathbb{P}^2$ we have a conic, and $\mathbb{P}^1$ can be embedded as this conic with the Veronese.
- In $\mathbb{P}^3$, $\mathbb{P}^1 \times \mathbb{P}^1$ can be embedded as the quadric by the Segre.
- We skip $\mathbb{P}^4$ for now.
- In $\mathbb{P}^5$ Grassmann and Plücker thought about it and we will consider it now.

You can think about $\mathbb{P}^n$ as

$$\mathbb{P}^n = \{L | L \subset \mathbb{C}^{n+1} \text{ is a 1-dim subspace}\}$$

Define the Grassmannian

$$\mathrm{Gr}(r, n) = \{S | S \subset \mathbb{C}^n \text{ is a } r\text{-dimensional subspace}\}$$

Can we express $\mathrm{Gr}(r, n)$ as a quotient? Let $\mathcal{M}$ be the space of $r \times n$ matrices, and let $\mathcal{U} \subset \mathcal{M}$ be the subset consisting of elements with rank r. We have $\mathcal{U}/\mathrm{Gl}_r \cong \mathrm{Gr}(r, n)$ (by the row space of the matrices).
A matrix has rank $< r$ if every $r \times r$ minor has det $= 0$. Let $\gamma \subset \{1, \ldots, n\}$ be a set with $|\gamma| = r$.

$$\begin{aligned} d_\gamma : \quad \mathcal{M} &\to \mathbb{C} \\ d_\gamma(\xi) &\mapsto \text{the determinant of the } \gamma\text{-minor of } \xi \end{aligned}$$

We have

$$U^C = \{\xi \in \mathcal{M} \mid \forall \gamma \subset \{1, \ldots, n\} \text{ with } |\gamma| = r : \ d_\gamma(\xi) = 0\} \subset \mathcal{M}$$

This is Zariski closed.
How can we think of $\mathrm{Gr}(r,n)$ as an algebraic projective variety? Let

$$\begin{aligned}\mathcal{P} : \mathrm{Gr}(r,n) &\to \mathbb{P}^{\binom{n}{r}-1} \\ \mathbb{C}^n \supset S &\mapsto [z_\gamma, \dots].\end{aligned}$$

Here, z_γ is the γ-minor of a matrix ξ_S that has the vectors of a basis of S as columns. The basis is not unique, so ξ_S is determined only up to Gl_r. For $\xi_S = g\hat{\xi}_S$, $g \in \mathrm{Gl}_r$ we have

$$(d_\gamma(\xi_s)), \dots) = (d_\gamma(g \cdot \hat{\xi}_S), \dots) = \det(g)(d_\gamma(\hat{\xi}_S), \dots)$$

so $[z_\gamma, \dots]$ is fixed up to a scalar! This fits perfectly with the homogeneous coordinates and the map is well-defined. It is called the *Plücker embedding* $\mathcal{P}$.
Example 10.1 Given an element ξ_S as

$$\begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{pmatrix}$$

we see that the $\mathrm{Gr}(2,4)$ is, in fact, 4-dimensional. We want to compute the Plücker $\mathcal{P}$. It is

$$\mathcal{P}(S) = [z_{12}, z_{13}, z_{14}, z_{23}, z_{24}, z_{34}] = [1, c, d, -a, -b, ad - bc]$$

The image satisfies the first Plücker equation

$$z_{12}z_{34} + z_{14}z_{23} - z_{13}z_{24} = 0$$

This is a rank 6 quadric in $\mathbb{P}^5 = \mathbb{P}^{\binom{4}{2}-1}$. We have a bijection of a full-rank quadric in $\mathbb{P}^5$ via the Plücker to $\mathrm{Gr}(2,4)$.

10.1 The incidence correspondence

Fix $r, n \in \mathbb{N}$.

Definition 10.2 The set

$$I = \{(S,p) | S \in \mathrm{Gr}(r,n),\ p \in \mathbb{P}(S) \subset \mathbb{P}^{n-1}\}$$

is called the *incidence correspondence*.

Theorem 10.3 $I \subset \mathrm{Gr}(r,n) \times \mathbb{P}^{n-1}$ is an algebraic subvariety.

Proof. As $\mathrm{Gr}(r,n)$ is an projective algebraic variety by the Plücker embedding, it is covered by affine opens U_J where $J \subset \{1, \dots, n\}$ with $|J| = r$. U_J corresponds to the r-dimensional subspaces of $\mathbb{C}^n$ where the determinant of the J-th minor is nonzero. The elements of U_J are therefore given (after reordering) by matrices of the form

$$\begin{pmatrix} 1 & & & x_{1,1} & \dots & x_{1,n-r} \\ & \ddots & & \vdots & & \vdots \\ & & 1 & x_{r,1} & \dots & x_{r,n-r} \end{pmatrix}$$

Therefore $U_I \cong \mathbb{C}^{r(n-r)}$.
To show that I is Z-closed it suffices to show that $I \cap U_J \times \mathbb{P}^{n-1}$ is Z-closed for all possible indices J. It is sufficient to check this for $J = \{1, \dots, r\}$; for all other indices, the procedure is the same. We have the variables $x_{11}, x_{12}, \dots, x_{1,n-r}, \dots, x_{r,n-r}$ in $\mathbb{C}^{r(n-r)}$ and homogeneous coordinates $[z_1, \dots, z_n]$. We want to find equations $F_1(x_{ij}, z_k), \dots,$ (where the x are "honest" coordinates, and the z are homogeneous, projective coordinates) that cut out $I_J \subset \mathbb{C}^{r(n-r)} \times \mathbb{P}^{n-1}$.
We get equations

$$\begin{aligned} F_1(x,z) &= z_{r+1} - x_{11}z_1 - x_{21}z_2 - \cdots - x_{r,1}z_r \\ F_2(x,z) &= z_{r+2} - x_{12}z_1 - x_{22}z_2 - \cdots - x_{r,2}z_r \\ &\vdots \\ F_{n-r}(x,z) &= z_n - x_{1,n-r}z_1 - \cdots - x_{r,n-r}z_r \end{aligned}$$

In conclusion $I \subset \mathrm{Gr}(r,n) \times \mathbb{P}^{n-1}$ is an algebraic subvariety. □

Proposition 10.4 Let $V \subset \mathbb{P}^3$ be a projective algebraic variety. Then the set of lines in $\mathbb{P}^3$ which meet V form an algebraic subvariety of $\mathrm{Gr}(2,4)$.

Proof. Let $\pi_1 : I \to \mathrm{Gr}(2,4), \pi_2 : I \to \mathbb{P}^3$ be the projection maps. Then $\pi_1\pi_2^{-1}(V)$ is the set of lines in $\mathbb{P}^3$ which intersect V. $\pi_2^{-1}(V)$ is closed. $\pi_1\pi_2^{-1}(V)$ is closed because the image of a projective variety under an algebraic map is closed (see theorem 14.3). □

Example 10.5 Example defining the Catalan numbers..

11 Irreducibility

Basically page 5 in Hartshorne.

12 Images of quasi-projective varieties under algebraic maps

Definition 12.1 Let X be a subset of a topological space. X is called *locally closed* if $X = U \cap V$ for U open and V closed. A *constructible set* is a finite union of locally closed subsets.

Theorem 12.2 Let X, Y be quasi-projective varieties. Let $f : X \to Y$ be algebraic. Then $\mathrm{Im}(f)$ is constructible.

13 Varieties defined by polynomials of equal degrees

Let $V \subset \mathbb{P}^n$ be a projective algebraic variety. $V = Z(F_1, \dots, F_k)$, where $F_i \in \mathbb{C}[z_0, \dots, z_n]$ is homogeneous of degree d_i. Let $d = \max\{d_1, \dots, d_n\}$. We want to find polynomials G_i all of degree d such that $V = Z(G_1, \dots, G_m)$. Construct them as follows:
Let

$$G_{i,j} := z_i^{d-d_j} F_j$$

where $0 \leq i \leq n, 1 \leq j \leq k$. Obviously $V \subset Z(\{G_{i,j} : 0 \leq i \leq n, 1 \leq j \leq k\}) =: V'$. Let $p \in V'$. We know that

$$\begin{aligned} z_0^{d-d_j} F_j(p) &= 0 \\ &\vdots \\ z_n^{d-d_j} F_j(p) &= 0, \end{aligned}$$

and it follows that $F_j(p) = 0$. $\Rightarrow V = Z(G_{1,1}, \ldots, G_{n,k})$.

Theorem 13.1 Let $V \subset \mathbb{P}^n$ be a projective algebraic variety, then we can find an embedding of V in $\mathbb{P}^N$ such that the image is defined by quadratic equations.

Proof. Let $V \subset \mathbb{P}^n$ be a projective variety. Assume that V is cut out by $F_1, \ldots, F_k$ all of degree d. Let

$$\begin{aligned} \nu_d : \mathbb{P}^n &\to \mathbb{P}^N \\ [z_0, \ldots, z_n] &\mapsto [z_0^d, \underbrace{\ldots}_{\text{all other monomials}}, z_n^d] \end{aligned}$$

be the d-Veronese map. The image of $\mathbb{P}^n$ is cut out by quadratic equations. $V \subset \mathbb{P}^n$ is cut out by $\{$quadrics that cut out $\nu_d(\mathbb{P}^n)$, linear terms corresponding to $F_1, \ldots, F_k\}$. □

14 Images of projective varieties under algebraic maps

The following will be useful:
Fact 14.1 If $G \in \mathbb{C}[Z_0, \ldots, Z_n]$ is a homogeneous polynomial of degree $d \geq 1$, then $U_G = \{\xi \in \mathbb{P}^n | G(\xi) \neq 0\}$ is an affine open, i.e. isomorphic to a closed subset of affine space.

Proof. We identify U_G with its image U under the d'th Veronese embedding $\nu : \mathbb{P}^n \to \mathbb{P}^N$. Let $V = \mathrm{Im}(\nu)$ Then G gives rise to a linear polynomial $L \in \mathbb{C}[Z_0, \ldots, Z_N]$ s.t. $U = U_L \cap V$. So U is a closed subset of U_L which, as L is linear, is isomorphic to $\mathbb{C}^N$. □

Fact 14.2 Any quasi-projective variety Y can be covered by finitely many affine open sets. This means that we can write $Y = \cup_{i=1}^N U_i$, where for each i, U_i is an open subset of Y isomorphic to a closed subset of the affine space $\mathbb{C}^m$.

Proof. There are $n \geq 1$, $F_1, \ldots, F_k, G_1, \ldots G_l \in \mathbb{C}[Z_0, \ldots, Z_n]$ homogeneous polynomials, s.t. $Y \subset \mathbb{P}^n$ is the set of points p, s.t. p is a zero of every F_i but not of every G_j. If we restrict our attention to the open set U_{G_i}, we see that $Y \cap U_{G_i} = V(F_1, \ldots, F_k) \cap U_{G_i}$. So it is clearly an open set of Y and at the same time a closed subset of U_{G_i} which can be viewed as a closed subset of affine space by the previous fact. □

Theorem 14.3 Let $f : X \to Y$ be an algebraic morphism between varieties, where X is projective and Y quasi-projective. Then

(A) $\mathrm{Im}(f)$ is Zariski-closed in Y.

(B) for $Z \subset X$ Zariski-closed, $f(Z)$ is Zariski-closed in Y.

Remark 14.4 Note that it suffices to prove either (A) or (B), as the other part is then easily implied.

Proof. We will prove Part (A) of the theorem.

More precisely, we will prove first that the graph of f (defined below) is Zariski-closed in $X \times Y$, and second that the projection $\Pi_Y : X \times Y \to Y$ is a closed map, whenever X is projective, Y quasi-projective, which can be reduced to the proof of the special case $X = \mathbb{P}^n$, $Y = \mathbb{C}^m$.

The first part works for any algebraic morphism$f : X \to Y$ between varieties. Define the *graph* of f,

$$\Gamma(f) := \{(x, y) \in X \times Y | f(x) = y\}$$

We want to show

$$\Gamma(f) \subset X \times Y \text{ is Zariski closed.} \tag{4}$$

As Y is a quasi-projective variety, we can view it as a subset $Y \subset \mathbb{P}^n$ for some n. Let $g : X \to \mathbb{P}^n$ be the composition of f and the embedding $i : Y \to \mathbb{P}^n$. Now it is clear that (4) will follow, if we can show instead

$$\Gamma(g) \subset X \times \mathbb{P}^n \text{ is Zariski closed.} \tag{5}$$

This can further be simplified. Consider the map

$$\begin{aligned} (g, id) : X \times \mathbb{P}^n &\to \mathbb{P}^n \times \mathbb{P}^n \\ (x, y) &\mapsto (g(x), y) \end{aligned}$$

This is algebraic by universal property of the product. Moreover, $\Gamma(g)$ is just the preimage of the diagonal

$$\Delta = \{(x, x) \in \mathbb{P}^n \times \mathbb{P}^n | x \in \mathbb{P}^n\}$$

under this map. Thus to prove (5) and thus finish the proof of (4), we have to prove that Δ is Zariski-closed. By taking a look at the Segre embedding

$$\begin{aligned} \mathbb{P}^n \times \mathbb{P}^n &\to \mathbb{P}^N \\ ([x_0, \ldots, x_n], [y_0, \ldots, y_n]) &\mapsto [x_i y_j]_{1 \le i,j \le n} \end{aligned}$$

we see that the Δ as a subset of $\mathbb{P}^N$ is defined by the equations

$$Z_{ij} = Z_{ji} \qquad 0 \le i < j \le n$$

together with the equations for the Segre embedding.

The second part takes a bit more work. We will start with reformulating the problem several times. More precisely, we will show how any of the following statements implies the one above it, and then prove the last:

$$\text{The projection } \Pi_Y : X \times Y \to Y \text{ is a closed map w.r.t. the Zariski topology.} \tag{6}$$

Claim 14.5

(i) (6) holds when X is a projective, Y a quasi-projective variety.

(ii) (6) holds when $X = \mathbb{P}^n$, Y a quasi-projective variety.

(iii) (6) holds for $X = \mathbb{P}^n$, $Y = U$, where U is a closed subset of affine space.

(iv) (6) holds for $X = \mathbb{P}^n$, $Y = \mathbb{C}^m$

(ii)⇒(i): As X is projective, $X \subset \mathbb{P}^n$ is a closed subset for some n. But then $X \times Y$ is closed inside $\mathbb{P}^n \times Y$. Now given $V \subset X \times Y$ closed, it is also closed in $\mathbb{P}^n \times Y$. So by (ii) its image under projection is closed in Y.

(iii)⇒(ii): Cover Y with finitely many affine opens $U_1, \dots, U_N$. Now note that any subset of Y is closed if and only if for $i = 1, \dots, n$ its intersection with U_i is closed in U_i. Also note that $\Pi_Y^{-1}(U_i) = X \times U_i$. Now let $V \subset X \times Y$ closed, $W \subset Y$ its image under projection. Then $V \cap X \times U_i$ is closed in $X \times U_i$. By (iii), its image under Π_Y, which is just $W \cap U_i$ is closed in U_i. This holds for any choice of i, so W is closed in Y.

(iv)⇒(iii): trivial

What is left is the proof of (iv).

Proof. Let $Z \subset \mathbb{P}^n \times \mathbb{C}^m$ be a closed subset. Z is the zero locus of polynomials

$$F_1, \dots F_k \in \mathbb{C}[Z_0, \dots, Z_n, Y_1, \dots, Y_m],$$

where F_i is homogeneous, when viewed as a polynomial in $Z_1, \dots Z_n$, say of degree d_i. We Want to find polynomial equations that describe $\Pi_Y(Z)$ in $\mathbb{C}^m$. It is easily seen that

$$\Pi_Y(Z) = \{(\xi_1, \dots, \xi_m) \in \mathbb{C}^n | \begin{matrix} F_1(Z_0, \dots, Z_n, \xi_1, \dots, \xi_m) \\ \vdots \\ F_k(Z_0, \dots, Z_n, \xi_1, \dots, \xi_m) \end{matrix} \text{ have a common zero in } \mathbb{P}^n\}$$

Let us take a step back, and reconsider the general question of when a set of homogeneous polynomials $G_1, \dots, G_l$ in the variables $Z_0, \dots Z_n$ has a common zero in projective space and when the opposite holds. Of the following statements any one is in an obvious way (by definition or by some basic result) equivalent to the next one.

1. $G_1, \dots, G_l$ have no common zero in $\mathbb{P}^n$.
2. The common zero locus of $G_1, \dots G_l$ in $\mathbb{C}^{n+1}$ is empty or equal to $(0, \dots, 0)$.
3. $\sqrt{(G_1, \dots, G_l)} = \begin{cases} (1) \\ (Z_0, \dots, Z_n) \end{cases}$
4. $\sqrt{(G_1, \dots G_l)} \supset (Z_0, \dots, Z_n)$
5. There is a $d \geq 1$ such that $(G_1, \dots, G_l)$ contains all monomials in $Z_0, \dots, Z_n$ of degree d.

Now if we apply the equivalence of 1. and 5. to our situation we immediately get

$$\begin{aligned} \Pi_Y(Z) &= \{\xi \in \mathbb{C}^m | \forall d \geq 1 : (F_1(Z, \xi), \dots, F_k(Z, \xi)) \text{ does not contain all monomials of degree } d.\} \\ &= \bigcap_{d=1}^{\infty} \{\xi \in \mathbb{C}^m | (F_1(Z, \xi), \dots, F_k(Z, \xi)) \text{ does not contain all monomials of degree } d.\} \\ &=: \bigcap_{d=1}^{\infty} W_d. \end{aligned}$$

As any intersection of closed sets is closed, we can consider one d at a time. For fixed d, all monomials of degree d are contained in the ideal $(F_1(Z, \xi), \dots, F_k(Z, \xi))$ if and only if they are contained in the subset of homogeneous elements of degree d, denoted by $(F_1(Z, \xi), \dots, F_k(Z, \xi))_d$, which is a vector space, and in particular a subspace of the space of homogeneous polynomials

of degree d, which is given by the monomial basis. So $(F_1(Z,\xi),\dots,F_k(Z,\xi))$ does not contain all monomials of degree d, if and only if

$$(F_1(Z,\xi),\dots,F_k(Z,\xi))_d \neq \mathbb{C}[Z_0,\dots,Z_n]_d.$$

$(F_1(Z,\xi),\dots,F_k(Z,\xi))_d$ is generated as a vector space by

$$\mathcal{S} := \{F_i(Z,\xi)Z^\alpha | 0 \leq i \leq n : d_i \leq d, |\alpha| = d - d_i\}$$

So the question is if these generate a space of dimension strictly lower than the number of monomials of degree d. By linear algebra this is the case iff for the matrix M where the columns are the coordinates of elements of $\mathcal{S}$ w.r.t. the monomial basis, every size-d minor has zero determinant. Note that this gives a finite set of polynomial equations in the entries of M, and this gives us polynomials in $\xi_1,\dots,\xi_m$, as every entry of M is a polynomial expression of those. W_d is the zero locus of the resulting polynomials, thus closed in $\mathbb{C}^m$. □

□

15 Bezout's Theorem

Last time we talked about Bézout's theorem, which in some sense is the analogue of the Fundamental Theorem of Algebra. In particular, given H_1 and H_2 in $\mathbb{P}^2$ where H_1 is of degree d_1 and H_2 of degree d_2, then H_1 and H_2 intersect d_1d_2 times, where points are counted with multiplicity.

One problem with this statement is that H_1 and H_2 can intersect in infinitely many points, say for example $H_1 = H_2 = (x_0)$. But at least we can restrict to the simple case: if $H_1 \cap H_2$ has finitely many points, then $|H_1 \cap H_2| = d_1d_2$, with multiplicities counted.

It may be the case that, at least locally, H_1 and H_2 intersect in two points, but after some parametrization they intersect in only one point. We want our definition of intersection to be independent of the parametrization, so we must somehow put some weight in each point to count for multiplicities. The way we calculate the weight of the point p_i is by considering the local ring at p_i and then taking the quotient by the equations of H_1 and H_2. The dimension of this quotient is equal to m_{p_i} and we get that

$$|H_1 \cap H_2| = \sum_{p_i \in H_1 \cap H_2} m_{p_i}.$$

15.1 The resultant

Let $p, q \in \mathbb{C}[x,y]$:

$$\begin{aligned} p &= \alpha_0 x^n + \alpha_1 x^{n-1}y + \cdots + \alpha_n y^n \\ q &= \beta_0 x^m + \beta_1 x^{m-1}y + \cdots + \beta_m y^m \end{aligned}$$

We want to find a polynomial R such that $R(\alpha_0,\dots,\alpha_n,\beta_0,\dots,\beta_m) = 0 \Leftrightarrow p, q$ have a common root. Let V_n be the vector space of homogeneous polymomials of degree n. We have $(x^n, x^{n-1}y,\dots,y^n)$ as a basis and $\dim V_n = n+1$. Here, denote $\mathbb{P}^n = \mathbb{P}(V_n)$. Let

$$\begin{aligned} f : \mathbb{P}^1 \times \mathbb{P}^{n-1} \times \mathbb{P}^{m-1} &\to \mathbb{P}^n \times \mathbb{P}^m \\ [\hat{l}, \hat{p}, \hat{q}] &\mapsto [\hat{l}\hat{p}, \hat{l}\hat{q}] \end{aligned}$$

f is an algebraic map (check this!), and therefore $\mathrm{Im}(f) \subset \mathbb{P}^n \times \mathbb{P}^m$ is Z-closed.
Let

$$\begin{aligned} L_{p,q} : V_{n-1} \otimes V_{m-1} &\to V_{n+m-1} \\ (k_1, k_2) &\mapsto k_1 q + k_2 p \end{aligned}$$

If there are $k_1 \neq 0, k_2 \neq 0$ such that

$$k_1 q + k_2 p = 0,$$

then $q | k_2 p$. This is equivalent to p and q having a common factor.
It follows that if p and q have no common factor, then $\ker f = 0$ and we have

$$p, q \text{ have no common factor} \Leftrightarrow \det(L_{p,q}) = 0$$

Definition 15.1 The matrix of $L_{p,q}$ in the standard basis $x^n, x^{n-1}y, \ldots, y^n$ is the resultant.

It is given by

$$L_{p,q} = \begin{pmatrix} \alpha_0 & \alpha_1 & \ldots & \ldots & \alpha_n & 0 & \ldots & 0 \\ 0 & \alpha_0 & \ldots & \ldots & \ldots & \alpha_n & \ddots & \vdots \\ \vdots & \ddots & \ddots & & & & \ddots & 0 \\ 0 & \ldots & 0 & \alpha_0 & \ldots & \ldots & \ldots & \alpha_n \\ \beta_0 & \beta_1 & \ldots & \ldots & \beta_m & 0 & \ldots & 0 \\ 0 & \beta_0 & \ldots & \ldots & \ldots & \beta_m & \ddots & \vdots \\ \vdots & \ddots & \ddots & & & & \ddots & 0 \\ 0 & \ldots & 0 & \beta_0 & \ldots & \ldots & \ldots & \beta_m \end{pmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}a\\a\\a\\a\end{matrix}}\right\} m \text{ rows} \\ \left.\vphantom{\begin{matrix}a\\a\\a\\a\end{matrix}}\right\} n \text{ rows} \end{matrix}$$

16 Pythagorean triples

In the following, we consider the solutions of the equation $x^2 + y^2 = z^2$. Over $\mathbb{C}$ this is a case which we have considered already, as the solutions form the conic $V(x^2 + y^2 - z^2)$ in $\mathbb{P}^2$. Thus we consider instead the solutions in $\mathbf{Q}$; over $\mathbf{Q}$ we may assume that $z = 1$ after dividing by z^2 (for $z = 0$ we only have the trivial solution). Geometrically, we want to determine the rational points on the unit circle $x^2 + y^2 = 1$. The geometric idea is rather simple; $(1, 0)$ is such a point, taking a line with rational slope through $(1, 0)$ we obtain another rational point on the unit circle, namely the intersection of the line with the unit circle. (All rational points on the unit circle are obtained in this way, since for every such point there is a unique line joining it to $(1, 0)$, which turns out to have rational slope.) Denoting the slope by s, the points on the line are of the form $(1, 0) + t(1, s) = (1 + t, ts)$. Plugging this into $x^2 + y^2 = 1$ yields $t = -2/(1 + s^2)$, so that the sought point has coordinates

$$\left(1 - 2/(1 + s^2), -\frac{2s}{1 + s^2}\right) = \left(\frac{s^2 - 1}{s^2 + 1}, -\frac{2s}{s^2 + 1}\right).$$

The family of these points are all rational points on the unit circle. In $\mathbb{P}^2$ this point has homogeneous coordinates $[s^2 - 1, 2s, s^2 + 1]$.

Now we consider $x^2 + y^2 = z^2$ as a Diophantine equation, considering only integral solutions; the simplest nontrivial solution being $(3, 4, 5)$.

17 The Riemann-Hurwitz formula

Let X and Y be surfaces of genus g and h, respectively. Furthermore let f be a branched covering map of degree d (i.e. every non-ramification point in X has exactly d preimages) which is ramified in only finitely many points. We denote these points by $b_1, \ldots, b_R$ and assume that at reach ramification only two sheets come together.

Theorem 17.1 (Riemann-Hurwitz formula)

$$2 - 2g = d(2 - 2h) - R \tag{7}$$

Proof. (sketch) We choose a triangulation of X such that $b_1, \ldots, b_R$ are corners of triangles. The preimage of this triangulation is (homeomorphic to) a triangulation of Y. Observe furthermore that all faces and edges have exactly d preimages under f. This also holds true for all vertices except the $b_1, \ldots, b_r$ which have $d - 1$ preimages each. Therefore, we have

$$\chi(Y) = d\chi(X) - R \tag{8}$$

It is also known that $\chi(X) = 2 - 2h$ and likewise $\chi(Y) = 2 - 2g$, giving

$$2 - 2g = d(2 - 2h) - R \tag{9}$$

□

This can be generalized for ramifications of arbitrary multiplicity. R is then equal to the sum of all multiplicities (greater than 1) minus the number of ramifications.

We can use the Riemann-Hurwitz formula to calculate the genus of varieties

Example 17.2 Let F be a homogeneous polynomial in three variables. If $F([0, 0, 1]) \neq 0$, we can canonically project $V(F)$ onto $\mathbb{P}^1$ using the projection map $\pi([x, y, z]) := [x, y]$. π is a branched covering of degree d because for fixed x and y, $F(x, y, z)$ is a polynomial in z of degree d. We also know that the genus of $\mathbb{P}^1$ is 0, giving $h = 0$. However, it is hard in general to determine the number and multiplicities of the ramifications and therefore R.

18 Points in projective space

Let $V = \{p_1, \ldots, p_r\} \subset \mathbb{P}^2$.

Proposition 18.1 V can be cut out by degree r polynomials.

Proof. Let

$$S = \{F | F = L_1 \ldots L_r, \text{ where } L_i \text{ is a line through } p_i\}$$

The polyomials in S all have degree r. We have that $V \subset Z(S)$. Let $q \in \mathbb{P}^2 \setminus V$. Then there a line through every point in V that misses q. Therefore $q \notin Z(S)$. $\Rightarrow V = Z(S)$. □

Example 18.2 A set of r distinct points in $\mathbb{P}^2$ cannot necessarily be cut out by polyomials of degree $r - 1$. For example let $L \subset \mathbb{P}^2$ be a line and let $p_1, \ldots, p_r$ be distinct points on the line. Let F be a polynomial of degree $r - 1$ which vanishes on $p_1, \ldots, p_r$. Then F vanishes everywhere on L.

Definition 18.3 The points $p_1, \dots, p_k \in \mathbb{P}^n$ are in *general linear position* if corresponding vectors $v_1, \dots, v_k \in \mathbb{C}^{n+1}$ are as linearly independent as possible. This means that for each subset S with $|S| \leq n+1$, $\dim \operatorname{span} S = |S|$.

Proposition 18.4 Let

$$\begin{aligned} \phi : \mathbb{P}^1 &\to \mathbb{P}^d \\ [s,t] &\mapsto [s^d, s^{d-1}t, \dots, t^d] \end{aligned}$$

be the d-Veronese. Let $\{p_0, \dots, p_d\} \subset \mathbb{P}^1$ be $d+1$ distinct points. Then $\{\phi(p_0), \dots, \phi(p_d)\}$ are in general linear position.

Proof. See exercise 3 on sheet 4. □

Definition 18.5 Let $P_0, \dots, P_n$ be a basis of the homogeneous polynomials of degree n. Then the map

$$\begin{aligned} \mathbb{P}^1 &\to \mathbb{P}^n \\ [s,t] &\mapsto [P_0(s,t), \dots, P_n(s,t)] \end{aligned}$$

is called a *rational normal curve*.

Example 18.6 The Veronese map from $\mathbb{P}^1$ is an example of a rational normal curve.

Theorem 18.7 Let $p_1, \dots, p_{n+3} \in \mathbb{P}^n$ be in general linear position. Then there is a unique rational normal curve in $\mathbb{P}^n$ passing through these points.

Proof. Choose a basis of $\mathbb{C}^{n+1}$ such that

$$\begin{aligned} p_1 &= [1, 0, \dots, 0] \\ p_2 &= [0, 1, \dots, 0] \\ &\vdots \\ p_{n+1} &= [0, 0, \dots, 1]. \end{aligned}$$

We have $p_{n+2} = [\alpha_0, \dots, \alpha_n]$. We have $\forall i \in \{0, \dots, n\} : \alpha_i \neq 0$, because else the points would not be in general linear position. We can rescale the basis vectors such that

$$p_{n+2} = [1, 1, \dots, 1].$$

We have $p_{n+3} = [\lambda_0, \dots, \lambda_n]$, where $\forall i \in \{0, \dots, n\} : \lambda_i \neq 0$ and $i \neq j \Rightarrow \lambda_i \neq \lambda_j$, because of the general linear position.
We are looking for a rational normal curve $\phi : \mathbb{P}^1 \to \mathbb{P}^n$ and points $\xi_1, \dots, \xi_{n+3} \in \mathbb{P}^1$ such that $\phi(\xi_i) = p_i$. Denote for $i \in \{1, \dots, n+1\} : \xi_i = [1, \xi_i]$.
Now $\phi(\xi_i) = [P_1(\xi_i), \dots, P_{n+1}(\xi_i)] = [0, \dots, 0, 1, 0, \dots, 0]$ implies that (setting $x = t/s$) $(x - \xi_i) \mid P_k$ for $k \neq i$ and therefore

$$P_i = \alpha_i \prod_{j \neq i} (x - \xi_j)$$

From $\phi([0,1]) = [1,\dots,1]$ we can infer that all α_i are equal (we can set them equal to 1). We have

$$\phi([1,0]) = [\prod_{i\neq 1}\xi_i,\dots,\prod_{i\neq n+1}\xi_i] = \left[\frac{1}{\xi_1},\dots,\frac{1}{\xi_{n+1}}\right] = [\lambda_1,\dots,\lambda_{n+1}]$$

and therefore we fix $\xi_i = \frac{1}{\lambda_i}$ which is possible, because $\lambda_i \neq 0$. □

19 Rational functions.

20 Tangent Spaces I

In Differential Geometry, tangent spaces, at least for smooth submanifolds, arise very naturally. The tangent space at a single point is best described as the collection of possible starting directions one can take when travelling from that point along the manifold. We want to develop a similar notion for algebraic varieties. For this, we will start with the definition for affine varieties and build from that towards a more general formulation.
Let $X \subset \mathbb{C}^n$ be an affine algebraic variety, $p \in X$

Definition 20.1 (First definition of Tangent Space)

1. We define $T_p(\mathbb{C}^n)$ to be the complex n-dimensional vector space generated by the abstract basis $\partial_1,\dots,\partial_n$.
2. For $f \in \mathbb{C}[X_1,\dots,X_n]$, $v = \sum_{i=1}^n v_i\partial_i \in T_p$, let
$$(D_v f)_p := \sum_{i=1}^n v_i \frac{\partial_i f}{\partial x_i}(p)$$
3. $T_p(X) := \{v \in T_p(\mathbb{C}^n) \mid \forall f \in \mathcal{I} : (D_v f)_p = 0\}$
is the tangent space of X at the point p.

Note that for $(D_v f)_p$ the usual properties of differentiation, such as linearity and the Leibniz rule, hold. We will also write $D_v f$ if there is no ambiguity concerning the basepoint.

Lemma 20.2 Let $f_1,\dots,f_k$ be generators of $\mathcal{I}(X)$. Then

$$T_p(X) = \{v \in T_p(\mathbb{C}^n) \mid \text{For } i = 1,\dots,k : (D_v f_i)_p = 0\}$$

Proof. Let $v \in T_p(\mathbb{C}^n)$ s.t. $D_v f_1 = \ldots = D_v f_k = 0$.
Then for $g = h_1 f_1 + \ldots + h_k f_k \in \mathcal{I}(X)$ we have

$$(D_v g)_p = (D_v \sum_{i=1}^n h_i f_i)_p = \sum_{i=1}^n (D_v h_i)_p f_i(p) + h_i(p)(D_v(f_i))_p = 0$$

As, for any i, by assumption $D_v f_i = 0$ and by definition of $\mathcal{I}(X)$: $f_i(p) = 0$. So $v \in T_p(X)$. This proves one inclusion. The other one is obvious.

□

Wen can always assume that up to a translation $p = (0, \ldots, 0)$. Then $T_p(X)$ is especially easy to compute:
Let $\mathcal{I} = (f_1, \ldots, f_k)$. We can write

$$\begin{aligned} f_1 &= \alpha_{11}X_1 + \alpha_{12}X_2 + \ldots + \alpha_{1n}X_n \ + \mathcal{O}(X^2) \\ f_2 &= \alpha_{21}X_1 + \alpha_{22}X_2 + \ldots + \alpha_{2n}X_n \ + \mathcal{O}(X^2) \\ &\vdots \\ f_k &= \alpha_{k1}X_1 + \alpha_{k2}X_2 + \ldots + \alpha_{kn}X_n \ + \mathcal{O}(X^2) \end{aligned}$$

Then the tangent space is given as the Kernel of the Matrix

$$A = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \ldots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \ldots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{n1} & \alpha_{n2} & \ldots & \alpha_{nn} \end{pmatrix}$$

It is possible to generalize the tangent space to arbitrary quasi-projective varieties, but for this a more abstract definition is neccessary. First we will show, that there is a more abstract way to define the tangent space of an affine variety.

Definition 20.3 (Second Definition of Tangent space) Let X be an affine variety, with coordinate ring $\Gamma(X)$. For $p \in X$ let

$$\mathfrak{m}_p = \{f \in \Gamma(X) \mid f(p) = 0\}$$

be the corresponding maximal ideal. Then

$$T_p(X) = (\mathfrak{m}_p/\mathfrak{m}_p^2)^\vee$$

The compatibility to our previous definition comes from the following
Claim 20.4 When using the definition of tangent space given in 20.1, we have a canonical isomorphism

$$T_p(X) \cong (\mathfrak{m}_p/\mathfrak{m}_p^2)^\vee$$

Proof. Suppose $X \subset \mathbb{C}^n$.The map

$$\begin{aligned} B : T_p(X) \times \mathbb{C}[X_1, \ldots, X_n] &\to \mathbb{C} \\ (v, f) &\mapsto (D_v f)_p \end{aligned}$$

By definition of the tangent space for every $v \in T_p(X)$ the linear map $B(v, \cdot)$ has kernel $\mathcal{I}(X)$. Thus by factoring we get a bilinear map $B : T_p(X) \times \Gamma(X) \to \mathbb{C}$, and by restricting its domain $B : T_p(X) \times \mathfrak{m}_p \to \mathbb{C}$. Now let $v \in T_p(X)$ and $f \in \mathfrak{m}_p^2$. Then $f = f_1g_1 + \ldots + f_kg_k$, for $f_i, g_i \in \mathfrak{m}_p, i = 1, \ldots, k$. Then by linearity of differentiation, Leibniz rule and as $f_i(p) = g_i(p) = 0$ for all i

$$B(v, f) = (D_v f)_p = \sum_{i=1}^{k} (D_v f_i)_p g_i(p) + f_i(p)(D_v g_i)_p = 0.$$

That means we can factor by $\mathfrak{m}_p^2$ which finally gives the bilinear:

$$B : T_p(X) \times \mathfrak{m}_p/\mathfrak{m}_p^2 \to \mathbb{C}.$$

This datum is equivalent to the linear map

$$\begin{aligned} L : T_p(X) &\to \left(\mathfrak{m}_p/\mathfrak{m}_p^2\right)^\vee \\ v &\mapsto (\overline{f} \mapsto (D_v f)_p) \end{aligned}$$

which we will now prove to be an isomorphism. Write $p = (\xi_1, \ldots, \xi_n)$ such that $\mathfrak{m}_p = (\overline{X}_1 - \xi_1, \ldots, \overline{X}_n - \xi_n)$. Assume $v \in T_p(X)$ such that for all $\overline{f} \in \mathfrak{m}_p$ $(D_v f)_p = 0$. Then in particular

$$v_i = (D_v(\overline{X}_i - \xi_i))_p = 0,$$

so $v = 0$. This proves injectivity. Now assume $\psi : \mathfrak{m}_p/\mathfrak{m}_p^2 \to \mathbb{C}$ is linear. Any $f \in \mathcal{I}(X)$ fulfills $f(p) = 0$ and hence can be written as

$$f = \sum_{i=1}^{n} (X_i - \xi_i) a_i + (\text{higher terms}).$$

As the image of f in $\Gamma(X)$ is zero, $\psi(\overline{f}) = 0$. At the same time $(\overline{X}_i - \xi_i)_{1 \le i \le k}$ span the $\mathbb{C}$-vector space $\mathfrak{m}_p/\mathfrak{m}_p^2$, and we have,

$$\psi(\overline{f}) = \psi(\sum_{i=1}^{n} (\overline{X}_i - \xi_i) a_i) = \sum_{i=1}^{n} \psi(\overline{X}_i - \xi_i) a_i.$$

Let $v_i = \psi(\overline{X}_i - \xi_i)$ and

$$v = \sum_{i=0}^{n} v_i \partial_i.$$

Then by what we have shown it follows that $v \in T_p(X)$ and that $L(v) = \psi$. This proves surjectivity of L. $\square$

We now want to generalize this definition in a straightforward way to arbitrary quasi-projective varieties. One possibility would be to choose for each point an affine open neighborhood and apply the second definition. However, we will use a more invariant definition, which clearly does not depend on a choice. The main problem is how to find the appropriate ring and maximal ideal so that we can do the same as in Def. 20.3. The solution comes with the following

Definition 20.5 For any quasi-projective variety X and $p \in X$, we define the ring of germs of algebraic functions of X at p as

$$\mathcal{O}_p := \{(U, f) \mid p \in U \subset X \text{ is open, } f \text{ is an algebraic function on } U.\}/\sim$$

Where the equivalence relation $\sim$ is defined as:

$$(U, f) \sim (V, g) \Leftrightarrow \exists W \subset U \cap V : W \text{ open and } f|_W = g|_W$$

It is an easy exercise that together with the induced addition and multiplication, this is indeed a local ring where the maximal ideal consists of the classes of functions that vanish at p.

Definition 20.6 (Third and last definition of the Tangent space) Let X be any quasi-projective variety and $p \in X$. Let $\mathcal{O}_p$ be the ring of germs of algebraic functions at p, $\mathfrak{m}_p$ its maximal ideal. Then the tangent space of X at p is

$$T_p(X) := \left(\mathfrak{m}_p/\mathfrak{m}_p^2\right)^\vee .$$

Again, we will show that this is compatible with our previous definitions:
Claim 20.7 Let X is a quasi-projective variety, $p \in X$. Let $T_p(X)$ be the Tangent space at p as in Def. 20.6. If $V \subset X$ is an affine open subset and $p \in V$, there is a canonic isomorphism between $T_p(V)$ as in Def. 20.3 and $T_p(X)$.

Proof.
We will use the following commutative algebra lemma which you can check for yourself:

Lemma 20.8 Let R be ring and $\mathfrak{m} \subset R$ a maximal ideal. Any $R/\mathfrak{m}$ vector space M is isomorphic as an R-module to its localization: $M \cong M \otimes_R R_\mathfrak{m}$

As restriction to an open neighborhood does not change the ring of functions, we can assume that $X = V$. Let $A = \Gamma(V)$ be the ring of functions of the affine variety V and $\mathfrak{m}_p$ the maximal ideal belonging to p. By exercise sheet 8, Ex. 2, we know that $\mathcal{O}_p \cong A_{\mathfrak{m}_p}$. Let $\mathfrak{m}_p$ the image of $\mathfrak{m}_p$ in $\mathcal{O}_K$. Now we consider the exact sequence of A-modules:

$$0 \to \mathfrak{m}_p^2 \to \mathfrak{m}_p \to \mathfrak{m}_p/\mathfrak{m}_p^2 \to 0.$$

Localization at $\mathfrak{m}_p$ gives:

$$0 \to \mathfrak{m}_p^2 \otimes A_{\mathfrak{m}_p} \to \tilde{\mathfrak{m}}_p \to \mathfrak{m}_p/\mathfrak{m}_p^2 \otimes A_{\mathfrak{m}_p} \to 0$$

Now you should note that $\mathfrak{m}_p^2 \otimes A_{\mathfrak{m}_p} = \tilde{\mathfrak{m}}_p^2$ and use the lemma to get:

$$\mathfrak{m}_p/\mathfrak{m}_p^2 \cong \tilde{\mathfrak{m}}_p/\tilde{\mathfrak{m}}_p^2.$$

Taking the dual on both sides yields the isomorphism of the tangent spaces. □

21 Tangent Space II

22 Blow-up

As a motivation for the blowup construction, consider the polar coordinates

$$\rho : \mathbb{R} \times [0, \pi]/\sim \longrightarrow \mathbb{R}^2, \quad \rho : (r, \phi) \mapsto (r\cos(\phi), r\sin(\phi)),$$

where $(r, 0) \sim (-r, \pi)$. Note that the domain is a bit unusual, but on $U = \{r \neq 0\}$, ρ is isomorphic to the classical polar coordinates defined on $\mathbb{R}^{>0} \times S^1$.

Note the characteristic feature of ρ: It is an isomorphism on $U = \{r \neq 0\}$ and contracts the circle $r = 0$ onto the zero-point. Polar coordinates are used to simplify calculations in calculus. We consider an analog construction in algebraic geometry, the blowup.

The blowup of $\mathbb{A}^2$ at 0 is defined by

$$\mathrm{Bl}_0\mathbb{A}^2 = \{(x,y),[u,v] \in \mathbb{A}^2 \times \mathbb{P}^1 \mid xv - yu = 0\}$$

together with the natural map

$$\rho : \mathrm{Bl}_0\mathbb{A}^2 \longrightarrow \mathbb{A}^2$$

given by projection on the first factor. It is easy to see that ρ is an isomorphism on $Bl_0\,\mathbb{A}^2 \setminus \{\rho^{-1}(0,0)\}$, while it contracts the central fiber $\rho^{-1}(0,0) \cong \mathbb{P}^1$ to the point $(0,0)$.

In fact, when considering the blowup construction over the real numbers $\mathbb{R}$, we obtain the motivating example above (up to an isomorphism; use $\mathbb{RP}^1 = S^1/\pm 1 = [0,\pi]/\sim$).

Then further discussion of the blowup, see Harris, Shafarevich, Hartshorne. Explicit computations are given in the solutions to sheet 9.

23 Dimension I

Topic is the equivalence of the definitions and several things about transcendence degree. Christoph recommended hungerford's algebra for more details.

24 Dimension II

25 Sheaves

Let X be a topological space. A *presheaf* of abelian groups (or rings, etc.) on X is a contravariant functor $\mathscr{F}$ from the poset category of open subsets of X to the category of abelian groups (rings, etc.). It is standard to denote $\mathscr{F}(U)$ by $\Gamma(U,\mathscr{F})$ (whose elements are called *sections* of $\mathscr{F}$ over U), and, for $V \subset U$, to denote the image of $s \in \mathscr{F}(U)$ under $\mathscr{F}(U) \to \mathscr{F}(V)$ by $s|_V$.

Roughly speaking, a sheaf is a presheaf whose sections are determined locally, and compatible sections can be patched together; it establishes relations between local and global properties of a space. Formally, a *sheaf* on X is a presheaf $\mathscr{F}$ on X such that for every open subset U of X and every open covering $\{U_i\}$ of U the following hold: (i) if $s \in \mathscr{F}(U)$ and $s|_{U_i} = 0$ for all i then $s = 0$; (ii) if $\{s_i\}$ is a family of sections $s_i \in \mathscr{F}(U_i)$ with $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for every i,j then there is a section $s \in \mathscr{F}(U)$ such that $s|_{U_i} = s_i$ for all i. (By (i) the section s must be unique.) Another way of putting this (giving a clue of how to generalize to sheaves with values in less concrete categories): we have an equalizer diagram

$$\mathscr{F}(U) \to \prod \mathscr{F}(U_i) \rightrightarrows \prod \mathscr{F}(U_i \cap U_j)$$

were the first map is given by $s \mapsto \{s|_{U_i}\}$ and the the other maps are $\{s_i\} \mapsto \{s_i|_{U_i \cap U_j}\}$ and $\{s_i\} \mapsto \{s_j|_{U_i \cap U_j}\}$. (Sometimes, for instance in analysis books, a different definition of sheaves is used, involving the *espace étalé*.)

Let $\mathscr{F}$ be a presheaf on X. Notice that if $x \in X$ is a point, the set of open neighborhoods U of x is directed under reverse inclusion; this gives a direct system $\mathscr{F}(U)$ (where the maps are the restrictions). The *stalk* of $\mathscr{F}$ at x, denoted $\mathscr{F}_x$, is the direct limit $\varinjlim \mathscr{F}(U)$ of this direct system. The construction (using the disjoint union, see exercise sheet 3 of the commutative algebra course) used to demonstrate the existence of the direct limit shows that we may think of $\mathscr{F}_x$ as having as elements equivalence classes $[(s,U)]$ with U an open neighborhood of x and $s \in \mathscr{F}(U)$ (germs of sections of $\mathscr{F}$ at x); $[(s,U)]$ and $[(t,V)]$ are considered as equal if there is an open neighborhood $W \subset U \cap V$ of x with $s|_W = t|_W$. If U is an open neighborhood of x, the canonical map $\mathscr{F}(U) \to \mathscr{F}_x$ belonging to the direct limit is by taking the germ at x,

$s \mapsto [(s, U)]$; these maps induce a map $\mathscr{F}(U) \to \prod_{x \in U} \mathscr{F}_x$ which is injective whenever $\mathscr{F}$ is a sheaf.

Example 25.1 If X is a variety (topological space, C^∞-manifold), then we have (in the obvious way) a sheaf of regular (continuous, C^∞-) functions on X. The reason that this is a sheaf is that the notion of a regular (continuous, differentiable) function is defined locally.

Example 25.2 The constant sheaf associated to an abelian group, the skyscraper sheaf.

Example 25.3 Let X and Y be spaces, U an open subset of X. A sheaf $\mathscr{F}$ on X induces one on U, denoted $\mathscr{F}|U$. If $f : X \to Y$ is a continuous map, the *direct image sheaf* $f_*\mathscr{F}$ of $\mathscr{F}$ is the sheaf on Y given by the composite of $\mathscr{F}$ and the functor from the category of open subsets of Y to the category of open subsets of X by taking the preimage under f.

A *morphism of (pre-)sheaves* is simply a functorial morphism. Every morphism $\phi : \mathscr{F} \to \mathscr{G}$ of presheaves on X gives naturally rise to notions such as its kernel, cokernel, image, and so on (see exercise sheet 11); also it induces (by the functoriality of $\varinjlim$, noticing that every morphism of presheaves gives a morphism of direct systems) for every $x \in X$ a map $\phi_x : \mathscr{F}_x \to \mathscr{G}_x$ of stalks (which is explicitly given by $\phi_x([(s, U)]) = [(\phi_U(s), U)]$). A sequence $\mathscr{F} \to \mathscr{G} \to \mathscr{H}$ of presheaves on X is exact iff $\mathscr{F}(U) \to \mathscr{G}(U) \to \mathscr{H}(U)$ is exact for all open subsets U of X.

However, the presheaf cokernel of a morphism of sheaves need not be a sheaf (but the presheaf kernel is one); this leads to sheafificiation (see exercise sheet 11). A sequence $\mathscr{F} \to \mathscr{G} \to \mathscr{H}$ of sheaves on X is exact iff it is exact at all stalks.

Example 25.4 The sequence of sheaves of abelian groups on $\mathbb{C}$

$$0 \to 2\pi i\mathbb{Z} \to \mathcal{O}_{\mathbb{C}}^{\mathrm{hol}} \xrightarrow{\exp} \mathcal{O}_{\mathbb{C}}^{\mathrm{hol},*} \to 0$$

is exact since it is exact at stalks (every nowhere vanishing holomorphic function is locally the exponential of some holomorphic function). The sequence of sections need, however, not be exact: for

$$0 \to 2\pi i\mathbb{Z} \to \mathcal{O}_{\mathbb{C}}^{\mathrm{hol}}(\mathbb{C}^*) \xrightarrow{\exp} \mathcal{O}_{\mathbb{C}}^{\mathrm{hol},*}(\mathbb{C}^*) \to 0$$

cannot be exact as $\exp : \mathcal{O}_{\mathbb{C}}^{\mathrm{hol}}(\mathbb{C}^*) \to \mathcal{O}_{\mathbb{C}}^{\mathrm{hol},*}(\mathbb{C}^*)$ is not surjective ($\mathrm{id}_{\mathbb{C}^*}$ is not in the image, as there is no global logarithm on $\mathbb{C}^*$).

26 Schemes I

Motivation for schemes: capturing nilpotents (varieties ignore nilpotents, e.g. $V(x) = V(x^2)$, the rings of algebraic functions are reduced); algebraic geometry over ground fields that are not algebraically closed, such as finite fields (the word on the street is that algebraic geometry over finite fields comes up in number theory, apparently this leads to some kind of unification of geometry and arithmetic).

A *ringed space* is a pair $(X, \mathcal{O}_X)$ consisting of a space X and a sheaf of rings $\mathcal{O}_X$ on X; $\mathcal{O}_X$ is said to be the *structure sheaf* of X. A morphism of ringed spaces is a pair $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ consisting of a continuous map $f : X \to Y$ and a morphism $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$ of sheaves on Y. A *locally ringed space* is a ringed space such that all stalks of the structure sheaf are local rings. A morphism of locally ringed spaces is a morphism of ringed spaces $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ such that for every $x \in X$ the canonical map $f_x^\sharp : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ explicitly given by $f_x^\sharp([(s, U)]) = [(f_U^\sharp(s), f^{-1}(U))]$ is a local homomorphism.

Example 26.1 If X is a space, then $(X, \mathcal{O}_X^{\mathrm{cont}})$ is a locally ringed space, since $\mathcal{O}_{X,x}^{\mathrm{cont}}$ is a local ring for every x (the maximal ideal being the germs of functions vanishing at x). Similiarly for $(X, \mathcal{O}_X^{\mathrm{sm}})$, X denoting a C^∞-manifold. In fact a C^∞-manifold of dimension n can be viewed as a locally $\mathbb{R}$-ringed second-countable Hausdorff topological space which is locally isomorphic (as a locally $\mathbb{R}$-ringed space) to $(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n}^{\mathrm{sm}})$.

Let A be a ring, consider the topological space Spec A; its definition and some of its topological properties were mentioned in the exercises of the commutative algebra course. Most notably Spec $A \approx$ Spec A^{red}, so that the topology on Spec A is insensitive to nilpotents; this is captured by the structure sheaf. The definition of the structure sheaf of Spec A is described in Hartshorne.

An *affine scheme* is a locally ringed space that is isomorphic to Spec A for some ring A. A *scheme* is a locally ringed space $(X, \mathcal{O}_X)$ which admits an open covering $\{U_i\}$ such that every $(U_i, \mathcal{O}_X|U_i)$ is an affine scheme.

Example 26.2 The affine scheme $\mathbb{C}^n$ is defined to be Spec $\mathbb{C}[x_1, \ldots, x_n]$.

27 Schemes II

Here he just proved Prop. 2.2. and Prop. 2.3. in Hartshorne. The main point is that the properties established in Prop. 2.2. are nice to have (in the lecture the definition of the structure sheaf of Spec A was motivated by these properties), and that Prop. 2.3. comes as a great relief since it is much easier to think about the (less-structured) category of rings than the highly-structured category of affine schemes. (Here a comparision with the fairly involved definition of a C^∞-manifold was drawn.)