Algebraic Geometry

D-MATH, FS 2014

Prof. Pandharipande

Exercise Sheet 3

1. a) Develop a theory of algebraic subvarieties of $\mathbb{CP}^n \times \mathbb{CP}^m$ using bihomogeneous polynomials. (For this you must define the Zariski topology and regular functions.)

b) Prove the Segre embedding $\mathbb{CP}^n \times \mathbb{CP}^m \longrightarrow \mathbb{CP}^{(n+1)(m+1)-1}$ is an algebraic morphism with respect to (a) — it is continuous in the Zariski topology and takes regular functions to regular functions.

c) Find equations for the image of the Segre embedding.

d) Prove that $\mathbb{CP}^n \times \mathbb{CP}^m$ is isomorphic to the image of the Segre embedding. In particular, show that the Segre embedding is bijective onto the image and the two have the same Zariski topology and the same notion of regular functions.

e) Prove that the projections $\mathbb{CP}^n \times \mathbb{CP}^m \longrightarrow \mathbb{CP}^n$ and $\mathbb{CP}^n \times \mathbb{CP}^m \longrightarrow \mathbb{CP}^m$ are algebraic maps.

2. Let $p_1, ..., p_{n+3}$ be $n+3$ points in general linear position in $\mathbb{CP}^n$. Prove that, there exists a rational normal curve (of degree n) which passes through the points $\{p_i\}$.

3. Prove that the image of the d-th Veronese embedding

$$\nu_d : \mathbb{CP}^n \longrightarrow \mathbb{CP}^{\binom{n+d}{n}-1}$$

is defined by quadratic equations (find the equations).

4. Prove that the variety defined by $x_0^2 + x_1^2 + x_2^2 = 0$ in $\mathbb{CP}^2$ is isomorphic to $\mathbb{CP}^1$.

5. Calculate the Euler characteristic and genus of the Fermat curve

$$X^d + Y^d + Z^d = 0$$

in $\mathbb{CP}^2$.

6. Calculate the number of lines in $\mathbb{CP}^3$ that meet 4 general skew lines.

7. Find 27 lines on the cubic surface $X^3 + Y^3 + Z^3 + W^3$ in $\mathbb{CP}^3$.

Due on March 14, 2014