ETH Zürich FS 2014
D-MATH
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Brownian Motion and Stochastic Calculus
Exercise Sheet 13

Exercise 13-1

For $i = 1, 2$, consider the BSDE

$$\begin{cases} dX_t^i = f_t^i(X^i, Z^i)dt + Z_t^i dW_t, \\ X_T^i = B^i. \end{cases} \tag{1}$$

Suppose that f^i is a Lipschitz generator and $B^i \in L^2(\mathcal{F}_T, P)$. If (X^i, Z^i) is a solution of (1) for $i = 1, 2$, prove that

$$\left\|X^1 - X^2\right\|_{\mathcal{R}^2}^2 + \left\|Z^1 - Z^2\right\|_{\mathcal{A}^2}^2 \leq C\Big(\left\|B^1 - B^2\right\|_{L^2(P)}^2 + \left\|f^1(X^2, Z^2) - f^2(X^2, Z^2)\right\|_{\mathcal{A}^2}^2\Big)$$

for some constant C.

Exercise 13-2

Let X be a Lévy process in $\mathbb{R}^d$ and $f_t(u) = E[e^{i\,u^{\mathrm{tr}}X_t}]$.

a) Show that X is stochastically continuous, i.e., $t \mapsto X_t$ is continuous in probability.

b) Show that $f_{t+s}(u) = f_t(u)f_s(u)$ for all $s, t \geq 0$ and $f_0(u) = 1$ for any $u \in \mathbb{R}^d$.

c) Use **b)** to show that $f_r(u) = f_1(u)^r$ for all *rational* $r \geq 0$.

d) Show that $t \mapsto f_t(u)$ is right-continuous and conclude that $f_t(u) = f_1(u)^t$ and that $f_t(u) \neq 0$ for all $t \geq 0$ and $u \in \mathbb{R}^d$.

e) Let $d = 1$. If $E[|X_1|] < \infty$, then $E[X_t] = tE[X_1]$ for all $t \geq 0$.

Exercise 13-3

a) Let N be a one-dimensional Poisson process and $(Y_i)_{i\geq 1}$ i.i.d. $\mathbb{R}^d$-valued random variables independent of N. We define the *compound Poisson process* by $X_t := \sum_{i=1}^{N_t} Y_i$. Show that X is a Lévy process and calculate its Lévy triplet.

b) Is there a Lévy process X such that X_1 is uniformly distributed on $[0, 1]$?

c) Let X and Y be both Lévy processes with respect to a filtration $(\mathcal{F}_t)$. Show that if $E[e^{i\,u^{\mathrm{tr}}X_t}e^{i\,v^{\mathrm{tr}}Y_t}] = E[e^{i\,u^{\mathrm{tr}}X_t}]\,E[e^{i\,v^{\mathrm{tr}}Y_t}]$ for all $u, v \in \mathbb{R}^d$ and $t \geq 0$, then X and Y are independent.