CHAPTER 3

Scalar conservation laws

In the previous section, we considered the scalar transport equation

$$U_t + a(x,t)U_x = 0. \tag{3.1}$$

This equation is linear as the velocity field a is a given function. However, most natural phenomena are nonlinear. In such models, the linear velocity field must be replaced with a field that depends on the solution itself. The simplest example of such a field is

$$a(x,t) = U(x,t).$$

Hence, the transport equation (3.1) becomes

$$U_t + UU_x = 0. \tag{3.2}$$

The transport equation (3.2) can be written in the *conservative form*

$$U_t + \left(\frac{U^2}{2}\right)_x = 0. \tag{3.3}$$

This is the inviscid *Burgers* equation. It serves as a prototype for *scalar conservation laws*, which in general take the form

$$U_t + f(U)_x = 0, \tag{3.4}$$

where U is the unknown and f is the flux function. Apart from Burgers' equation, scalar conservation laws arise in a wide variety of models. We consider a couple of examples below.

Traffic flow model. For simplicity, consider a one-dimensional highway and denote the density of cars (number of cars per square meter) as $U(x,t)$. Assume that the cars are moving at a *macroscopic* velocity (the speed of a traffic column) $V(x,t)$. A simple requirement of conservation of the number of cars lead to the following equation:

$$U_t + (UV)_x = 0. \tag{3.5}$$

The velocity V remains to be modeled. One very simple model is based on a couple of observations. First, there exists a maximum velocity at which an individual car can drive, for example specified by the speed limit. Second, the velocity of cars is inversely proportional to the car density. If there are a large number of cars, each individual driver will drive slowly. However, on a remote stretch of the highway, each driver speeds up. These simple observations are combined to yield the velocity

$$V = V_{\max}(1-U),$$

where $V_{\max}$ is the maximum velocity for the cars. We use the convention that the maximum density or *road carrying capacity* is 1. Hence, the traffic flow equation is

$$U_t + \big(V_{\max}U(1-U)\big)_x = 0. \tag{3.6}$$

Enhanced oil recovery. Oil is generally found in sub-surface reservoirs, inside permeable rocks. The primary stage of oil recovery consists of drilling into the rocks and extracting oil by applying pressure. Only 20 to 30 percent of the available oil can be extracted in this manner. The secondary stage of oil recovery consists of injecting water into the rock bed. The water displaces the oil (as water is heavier) and the oil can then be extracted. This complex process is modeled by using two-phase flow (water and oil) in a porous media (rock).

For simplicity, we assume that the reservoir is one-dimensional. The quantities of interest are the oil and water volume fractions or *saturations* S^o and S^w, respectively. Being volume fractions, they satisfy

$$S^o + S^w \equiv 1. \tag{3.7}$$

Furthermore, the phases evolve according to the conservation laws

$$\begin{cases} S^o_t + V^o_x = 0 \\ S^w_t + V^w_x = 0. \end{cases} \tag{3.8}$$

The phase velocities V^o, V^w are modeled by Darcy's law:

$$\begin{cases} V^o = -\lambda^o \frac{dP^o}{dx} \\ V^w = -\lambda^w \frac{dP^w}{dx}, \end{cases} \tag{3.9}$$

where λ and P are the phase mobility and the phase pressure, respectively. In the above constitutive relation, we have neglected the role of gravity. Furthermore, we can assume that there is no capillary pressure:

$$P^o = P^w.$$

Adding the phase saturation equations (3.8) for each phase and using the requirement (3.7), we obtain

$$(V^o + V^w)_x \equiv 0 \qquad \Rightarrow \qquad V^o + V^w = q,$$

for some constant q called the total flow rate. Substituting Darcy's law (3.9) in the above identity and using $P^w = P^o = P$, we obtain

$$\frac{dP}{dx} = -\frac{q}{\lambda^o + \lambda^w}.$$

Applying this identity in the evolution of the oil saturation (3.9) and (3.8) yields

$$S^o_t + \left(\frac{q\lambda^o}{\lambda^w + \lambda^o} \right)_x = 0. \tag{3.10}$$

The mobilities generally take the form

$$\lambda^o = (S^o)^2, \qquad \lambda^w = (S^w)^2 = (1 - S^o)^2.$$

Hence, the evolution of the oil saturation is governed by the scalar conservation law

$$S^o_t + \left(\frac{q(S^o)^2}{(S^o)^2 + (1 - S^o)^2} \right)_x = 0. \tag{3.11}$$

The above examples demonstrate that scalar conservation laws do occur in many interesting models in physics and engineering. Furthermore, the shape of the flux function f in (3.4) can be very general. Note that it is convex for Burgers' equation, concave for the traffic flow problem (3.6) and is neither convex nor concave (contains inflection points) for the oil reservoir equation (3.11).

In this section, we embark on a systematic study of scalar conservation laws (3.4) from a theoretical perspective.

3.1. Characteristics for Burger's equation

We start with Burgers' equation (3.3) and attempt to construct solutions to the initial value problem associated with it. As for the linear transport equation (3.1), we will use the method of characteristics for this purpose. Since (3.2) and (3.3) are equivalent whenever U is smooth, the characteristics $x(t)$ for Burgers' equation are given by

$$\begin{aligned} x'(t) &= U(x(t), t) \\ x(0) &= x_0. \end{aligned} \tag{3.12}$$

Note that these characteristics are different from the linear case (2.4) in that the velocity depends on the solution. We consider initial data

$$U_0(x) = \begin{cases} U_l & \text{if } x < 0 \\ U_r & \text{if } x > 0. \end{cases} \tag{3.13}$$

Data of this form is quite simple and consists of constants separated by a discontinuity at the origin. The initial value problem for a conservation law (3.4) with initial data of the form (3.13) is called a *Riemann problem*.

By definition, the solution U is constant along characteristics, that is, $U(x(t), t) = U_0(x_0)$. Therefore, the solution of (3.12), (3.13) in constant parts of U_0 is

$$x(t) = U_0(x_0)t + x_0.$$

Let $U_l = 1$ and $U_r = 0$ in (3.13). For $x_0 < 0$ the characteristics have velocity $U_0(x_0) = 1$, whereas for $x_0 > 0$ they have velocity 0; see Figure 3.1. We see that the characteristics intersect almost instantaneously. As observed in the last section, the solution should be constant (in time) along the characteristics. What happens to the solution when the characteristics start to intersect? How can the solution be defined in this case? Adding nonlinearity completely changes the situation from the linear case.

Is the intersection of characteristics on account of discontinuous data (3.13)? Can using smooth data lead to non-intersecting characteristics? It turns out that even smooth initial data can lead to the intersection of characteristics after a small time interval. Consider the visual example in Figure 3.2.

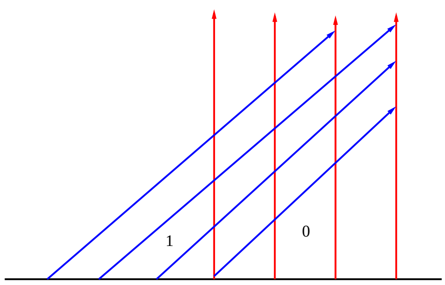
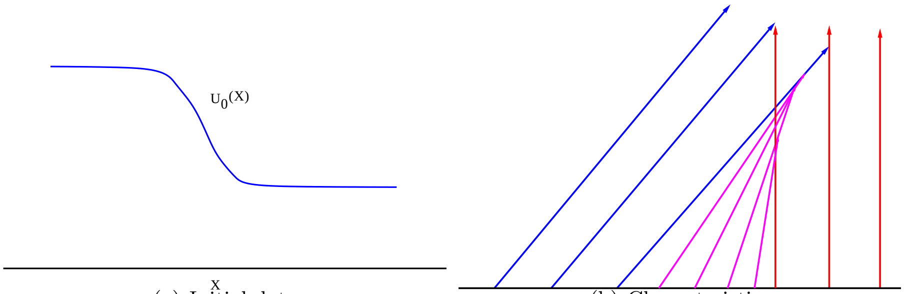


FIGURE 3.1. Characterstics intersecting for the Riemann problem (3.13) with $(U_l, U_r) = (1, 0)$.



(a) Initial data (b) Characteristics

FIGURE 3.2. Characteristics can even intersect for smooth initial data.

Exercise 3.1. *Let $U_0(x)$ be differentiable with at least one point x such that $U_0'(x) < 0$. Show that the solution to Burgers' equation with initial data U_0 will develop a discontinuity at time*

$$t_{\min} = -\frac{1}{\min_{x\in\mathbb{R}} U_0'(x)}.$$

(Hint: Start with the ansatz that two characteristics $x(t)$ and $\tilde{x}(t)$ intersect at some time t.)

The strange behavior of characteristics indicates that smooth solutions cannot be obtained for the conservation law (3.4), even when the initial data is smooth. Consider the initial data

$$U_0(x) = \sin(\pi x)$$

in the interval $[-1, 1]$. A heuristic interpretation of the characteristic equation (3.12) is that the solution at each point x moves with the velocity $U_0(x)$. Hence, the method of characteristics imply that the solution behaves as shown in Figure 3.3. The wave compresses in one part and stretches in another. In particular, the solution can be multi-valued. This is another indication that smooth solutions of (3.4) do not exist. A formal calculation by differentiating (3.2) with respect to x yields

$$V_t + UV_x = -V^2, \tag{3.14}$$

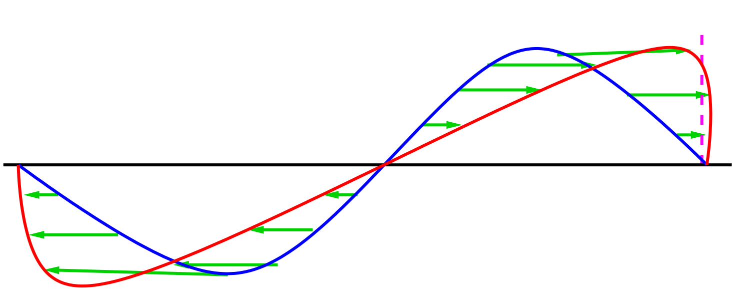


FIGURE 3.3. Smooth initial data leading to multi-valued solution.

where $V = U_x$. Hence, along the characteristics $x(t)$ given by (3.12), V varies as

$$\frac{d}{dt}V(x(t),t) = -V^2(x(t),t).$$

This is a ODE with quadratic nonlinearity and it is well known that the resulting solution V can blow up in finite time. Hence, the spatial derivative of the solution to Burgers' equation can blow up, even if the initial derivative is very small. This derivative blowup suggests that smooth solutions to (3.4) may not exist.

3.2. Weak solutions

The previous section demonstrates that smooth or classical solutions of the conservation law (3.4) may not exist. However, these models arise in physics and so *some* form of solution does exist. This type of solution is a *weak solution*. To motivate the definition of weak solutions, assume for the moment that smooth solutions of (3.4) exist and multiply both sides by a smooth test function $\varphi \in C_c^1(\mathbb{R} \times \mathbb{R}_+)$. The space C_c^1 is the space of all continuously differentiable functions with compact support, that is, the functions vanish outside a compact subset of the domain. Using integration by parts, (3.4) reduces to

$$\int_{\mathbb{R}\times\mathbb{R}_+} U\varphi_t + f(U)\varphi_x \; dxdt + \int_{\mathbb{R}} U_0(x)\varphi(x,0)\; dx = 0. \tag{3.15}$$

This identity hold trues for all test functions φ. We base the definition of weak solution for (3.4) on the above identity.

Definition 3.2 (Weak solution)**.** *A function $U \in L^1(\mathbb{R}\times\mathbb{R}_+)$ is a* weak solution *of* (3.4) *if the identity* (3.15) *holds for all test functions $\varphi \in C_c^1(\mathbb{R}\times\mathbb{R}_+)$.*

Note that the identity (3.15) is well-defined as long as $U \in L^\infty(\mathbb{R}\times\mathbb{R}_+)$.

Exercise 3.3. *Show that if a weak solution U of* (3.4) *is also differentiable (so $U \in C^1(\mathbb{R}\times\mathbb{R}_+)$), then U satisfies* (3.4) *point-wise. Hence, the class of weak solutions contains, but is not restricted to, classical solutions.*

The concept of weak solutions is highly non-standard. Our usual understanding of solutions of PDEs is classical – the solutions must be differentiable functions. However, weak solutions are not necessarily differentiable, not even continuous. This implies that the solutions can contain discontinuities. These discontinuities appear in nature as *shock waves.*

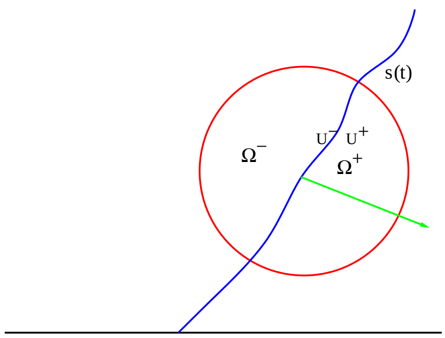


FIGURE 3.4. What happens across a shock?

3.2.1. The Rankine-Hugoniot condition. As we will soon find out, shock waves in weak solutions cannot be arbitrary curves in the x-t-plane, but must satisfy certain conditions. Assume that we are given a weak solution U consisting of two smooth regions, separated by a shock wave, as depicted in Figure 3.4. Let the shock wave be defined by the curve $x = \sigma(t)$.

Let φ be a test function with support in Ω. We assume that $U \in C^1(\Omega^-)$ and $U \in C^1(\Omega^+)$; see Figure 3.4. Integrating (3.15) by parts and using the compact support of the test function, we get

$$\begin{aligned}\int_\Omega U\varphi_t + f(U)\varphi_x \, d\Omega &= \int_{\Omega^+} U\varphi_t + f(U)\varphi_x \, d\Omega + \int_{\Omega^-} U\varphi_t + f(U)\varphi_x \, d\Omega \\ &= -\int_{\Omega^+} (U_t + f(U)_x)\varphi \, d\Omega + \int_{\partial\Omega^+} (U^+(t)\nu^t + f(U^+(t))\nu^x)\varphi \, d\Omega \\ &\quad - \int_{\Omega^-} (U_t + f(U)_x)\varphi \, d\Omega + \int_{\partial\Omega^-} (U^-(t)\nu^t + f(U^-(t))\nu^x)\varphi \, d\Omega \\ &= 0.\end{aligned}$$

Here, $U^+(t)$ and $U^-(t)$ are the trace values of U on the right and left of the discontinuity σ, and ν is the unit outward normal of σ (see Figure 3.4). We have

$$(\nu^t, \ \nu^x) = (-s(t), \ 1),$$

where $s(t) = \sigma'(t)$ is the speed of the shock curve. Since U is smooth in Ω^- and Ω^+, the equation (3.4) is satisfied point-wise. Therefore, the above identities imply that

$$\int_{\Omega^- \cup \Omega^+} \underbrace{\big(U_t + f(U)_x\big)}_{=\,0} \varphi \, d\Omega + \int_{\partial\Omega} \Big(s(t)\left(U^+(t) - U^-(t)\right) - \big(f(U^+) - f(U^-)\big)\Big)\varphi \, d\Omega = 0.$$

Since φ is an arbitrary test function, the integrand of the remaining integral must be identically equal to zero. Hence, the shock speed must satisfy

$$s(t) = \frac{f(U^+(t)) - f(U^-(t))}{U^+(t) - U^-(t)}. \tag{3.16}$$

This condition is called the *Rankine-Hugoniot* condition.

3.2.2. Solutions to Riemann problems. Consider Burgers' equation (3.3) with the Riemann problem (3.13) with $U_l = 1$ and $U_r = 0$. We recall that the characteristics intersected in this case and a smooth solution couldn't be constructed. We construct a weak solution that consists of two constant states U_l and U_r, separated by a shock moving at a speed given by the Rankine-Hugoniot condition (3.16),

$$s(t) = \frac{U_r^2 - U_l^2}{2(U_r - U_l)} \equiv \frac{1}{2}.$$

Hence, the weak solution takes the form

$$U(x,t) = \begin{cases} 1 & \text{if } x < \frac{1}{2}t \\ 0 & \text{if } x > \frac{1}{2}t. \end{cases} \tag{3.17}$$

It is easy to check that (3.17) satisfies (3.15). The structure of the solution (see Figure 3.5) shows that the characteristics *flow into* the shock. As a consequence, there are characteristics covering all points in the plane, and for each point we can trace a characteristic back to the initial data. Hence, the entire solution is prescribed by the initial data.

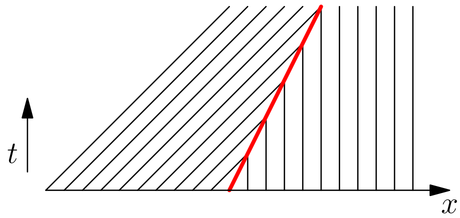


FIGURE 3.5. Characteristics for the Riemann problem (3.17).

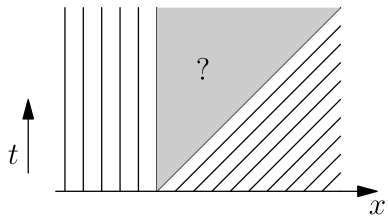
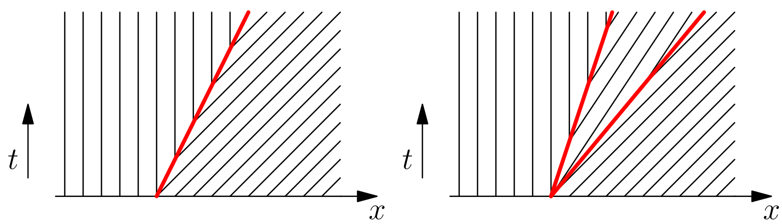


FIGURE 3.6. Characteristics for the Riemann problem (3.18).

Next, we consider another Riemann problem with $U_l = 0$ and $U_r = 1$. If we follow characteristics emanating from the x-axis, as for the previous problem, we now get an area without characteristics; see Figure 3.6. The "missing" information in this area may be "filled" in several ways. Using the Rankine-Hugoniot condition, we find that one possible weak solution is given by

$$U(x,t) = \begin{cases} 0 & \text{if } x < \frac{1}{2}t \\ 1 & \text{if } x > \frac{1}{2}t, \end{cases} \tag{3.18}$$

see Figure 3.7(a) Note that this solution has one shock curve, drawn in red in the figure. However, this solution is not the only possible weak solution. By adding



(a) Solution (3.18) (b) Solution (3.19)

FIGURE 3.7. Characteristics for different weak solutions the Riemann problem (3.18).

an intermediate state with value, say, $U_m = \frac{2}{3}$ and using the Rankine-Hugoniot condition, we get the weak solution

$$U(x,t) = \begin{cases} 0, & \text{if } x < \frac{1}{3}t \\ \frac{2}{3} & \text{if } \frac{1}{3}t < x < \frac{5}{6}t \\ 1 & \text{if } x > \frac{5}{6}t. \end{cases} \tag{3.19}$$

The characteristics are shown in Figure 3.7(b). In a similar manner one may construct arbitrarily many weak solutions by using the Rankine-Hugoniot condition (3.16) with different intermediate states.

This problem of non-uniqueness is implicit in the definition of weak solutions. These solutions are not necessarily unique, and therefore some extra conditions need to be imposed. For finding these extra criteria, we observe that characteristics for both (3.18) and (3.19) *flow out* from the shock (see Figure 3.7). This is in contrast to the solution (3.17) where the characteristics flow *into* the shock (see Figure 3.5). Characteristics represent the flow of information. For an evolution equation the information should always flow from the initial data. This is clearly the case for the weak solution (3.17). However in the case of weak solutions (3.18) and (3.19), information seems to be created at the shock.

This heuristic requirement, that information is taken from the initial data and is not created at a shock, can be expressed in terms of conditions on the characteristics across a shock. Let $U^-(t), U^+(t)$ be the states on either side of a shock with speed $s(t)$. The requirement that characteristics for Burgers' equation flow into the shock and information is taken from the initial line can be enforced by the condition

$$U^-(t) > s(t) > U^+(t). \tag{3.20}$$

It is simple to generalize (3.20) to the general scalar conservation law (3.4) for convex f:

$$f'(U^-(t)) > s(t) > f'(U^+(t)). \tag{3.21}$$

This is the *Lax entropy condition.*

Consider the conservation law (3.4) with a convex flux function and Riemann data (3.13). It is easily shown that

$$U(x,t) = \begin{cases} U_l & \text{if } x < st \\ U_r & \text{if } x > st, \end{cases} \tag{3.22}$$

where the shock speed s is defined by the Rankine-Hugoniot condition, is a weak solution of (3.4). Now, there are two cases: either $U_l > U_r$, or $U_l < U_r$. It turns out that the Lax entropy condition excludes (3.22) as a solution in the latter case, but not in the former:

Exercise 3.4. *Assume that f is strictly convex and that $U_l > U_r$. Show that* (3.22) *is a weak solution that satisfies the entropy condition* (3.21)*. Similarly, if $U_l < U_r$, show that* (3.22) *is a weak solution, but does* not *satisfy Lax' entropy condition.*

It turns out that in the latter case, where $U_l < U_r$, a continuous (but not necessarily differentiable) solution exists.

3.2.3. Rarefaction waves. For the remainder of this section, assume that the flux function f is strictly convex. In order to construct a continuous solution to (3.4), we note that replacing x, t by $\lambda x, \lambda t$ keeps the equation invariant, in the sense that a solution of one is a solution of the other. Hence, it is natural to assume *self-similarity* – that solutions only depend on the ratio x/t:

$$U(x,t) = V\left(x/t\right). \tag{3.23}$$

Define the similarity variable $\xi = x/t$. We substitute the ansatz (3.23) into (3.4) and use the chain rule repeatedly to obtain

$$\begin{aligned} V_t + f(V)_x &= V(\xi)_t + f'(V(\xi))V(\xi)_x \\ &= V_\xi \xi_t + f'(V(\xi))V_\xi \xi_x \\ &= -\frac{x}{t^2}V_\xi + f'(V(\xi))\frac{1}{t}V_\xi = 0 \\ \Rightarrow \qquad \left(f'(V(\xi)) - \frac{x}{t}\right)V_\xi = 0. \end{aligned}$$

In the nontrivial case of $V_\xi \neq 0$, the above identity and the fact that f' is strictly increasing (recall that f is assumed to be strictly convex) leads to the expression

$$V(x/t) = (f')^{-1}(x/t). \tag{3.24}$$

A self-similar solution of this form is called a *rarefaction wave*.

The rarefaction wave can be employed to construct weak solutions for conservation laws. Consider the Riemann problem (3.4), (3.13). If $U_l < U_r$, then the weak solution is given by

$$U(x,t) = \begin{cases} U_l & \text{if } \ x \leqslant f'(U_l)t \\ (f')^{-1}(x/t) & \text{if } \ f'(U_l)t < x \leqslant f'(U_r)t \\ U_r & \text{if } \ x > f'(U_r)t. \end{cases} \tag{3.25}$$

Clearly (3.25) is a weak solution that satisfies Lax' entropy condition (3.21). For the particular case of Burgers' equation with Riemann data $U_l = 0$ and $U_r = 1$, the solution (3.25) is shown in Figure 3.8. Note how the characteristics are parallel to the rarefaction wave and contrast this to Figure 3.7.

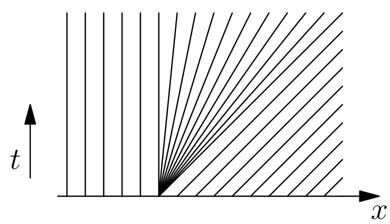


FIGURE 3.8. The rarefaction solution (3.25)

We now have a recipe to construct weak solutions for the Riemann problem (3.13) for a conservation law (3.4) with a strictly convex f. The solution depends on whether $U_l < U_r$ or $U_l > U_r$. If $U_l > U_r$, then the entropy satisfying weak solution (3.22) consists of a shock between the two states. If $U_l < U_r$, then the weak solution (3.25) consists of the two states, separated by a rarefaction wave.

In both cases, the wave speed is bounded in absolute value by the maximum of $|f'(U_l)|$ and $|f'(U_r)|$.

It remains to solve the Riemann problem when the flux is not strictly convex.

3.2.4. Solutions to the Riemann problem in the general case. We need to generalize the entropy condition (3.20) when the flux is no longer strictly convex. This is done by the following condition.

Definition 3.5 (Oleinik entropy condition)**.** *Let U be a weak solution of* (3.4)*, and let U_l, U_r be the trace values at a shock with speed s. (All of these quantities depend on t, but we drop the time dependence for notational convenience.) The solution satisfies the* Oleinik entropy condition *if*

$$s \leqslant \frac{f(k) - f(U_l)}{k - U_l}$$

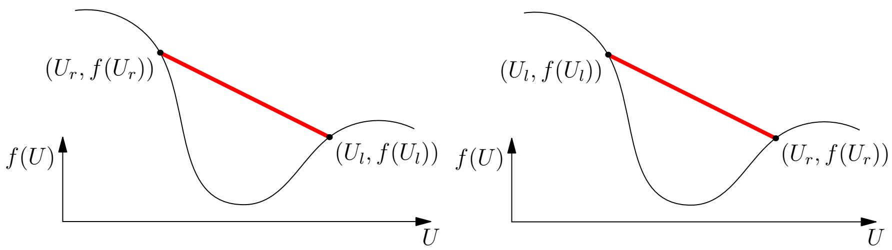
for all k between U_l and U_r.

Exercise 3.6. *Use the Rankine-Hugoniot condition to show that U satisfies the Oleinik entropy condition if and only if*

$$\frac{f(k) - f(U_r)}{k - U_r} \leqslant s$$

for all k between U_l and U_r.

A geometric interpretation of the above conditions is shown in Figure 3.9. It amounts to a convexity condition that the straight line joining $(U_l, f(U_l))$ and $(U_r, f(U_r))$ must lie below the graph of the function f between these points when $U_l < U_r$, and should lie above the graph if $U_l > U_r$. Clearly, the condition (3.20) for convex fluxes is a special case for this condition.



(a) Admissible (b) Inadmissible

FIGURE 3.9. Admissible and inadmissible shocks under the Oleinik entropy condition.

We have the following recipe for constructing a weak solution for the Riemann problem (3.13) for the conservation law (3.4), that satisfies the Oleinik entropy condition. Assume that the flux f is not convex but has finitely many inflection points (see Figure 3.10). Without loss of generality, we may assume that $U_l < U_r$. In order to satisfy the Oleinik entropy condition, we have to consider the *lower convex envelope* f_c of f between U_l and U_r. The lower convex envelope of a function f is the largest convex function (largest in the point-wise sense) that is everywhere

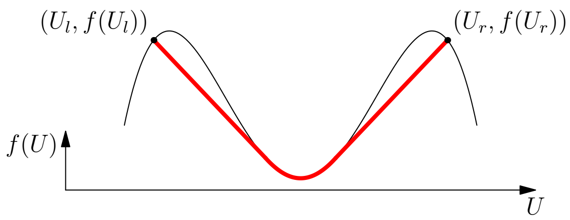


FIGURE 3.10. The solution of the Riemann problem with a non-convex flux. The lower convex envelope is the thick red curve. Solutions are constructed as shocks, followed by rarefactions

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smaller than or equal to f (see Figure 3.10). Analogously, the upper concave envelope of f is the smallest concave function that is larger than or equal to f.

The domain $[U_l, U_r]$ is divided into two sets of regions, one in which $f_c = f$ and another with $f_c \neq f$. In the second region, f_c is affine. The strategy for constructing an entropy solution is to join U_l and U_r by rarefaction waves and shocks. Shocks are used in the affine region and rarefactions in the complement. The solution of the Riemann problem is then (3.25) with f replaced by f_c. An illustration is provided in Figure 3.10.

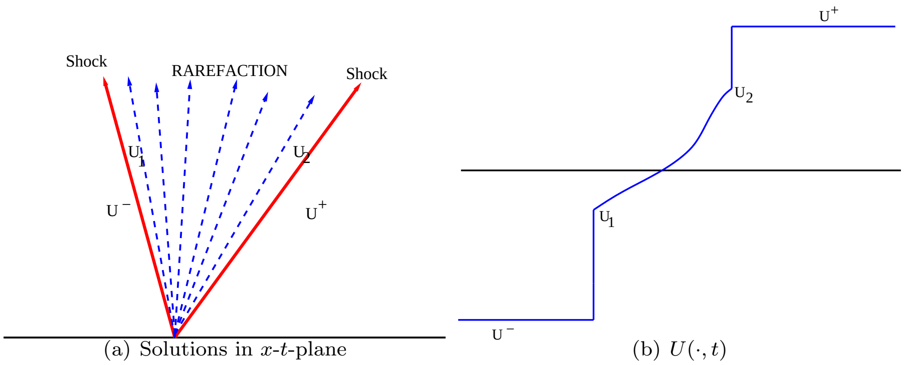

FIGURE 3.11. Entropy solutions for the Riemann problem with a non-convex flux. Left: Solutions in space-time. Right: A snapshot of the solution illustrating compound shocks.

When $U_l > U_r$, the upper concave envelope can be analogously used. Details of this construction can be obtained from [**1**]. A wave consisting of rarefaction, followed immediately by a shock (or vice versa) is termed a *compound shock* (see Figure 3.11).

3.3. Entropy solutions

The first step in analyzing solutions to the Cauchy problem for the conservation law (3.4) is to generalize the point-wise Oleinik entropy condition at a shock. This is the main focus of this section.

We consider the following *viscous* approximation to the scalar conservation law:

$$U_t^\varepsilon + f(U^\varepsilon)_x = \varepsilon U_{xx}^\varepsilon, \tag{3.26}$$

where $\varepsilon > 0$ is a small parameter. Note that (3.26) is a perturbation of the scalar conservation law. The second-order term U_{xx} is termed the *viscous* term. Adding the viscous term turns the conservation law into a convection-diffusion equation. Such equations are similar to the heat equation (1.8), and as for the heat equation, the solutions to (3.26) are smooth, in fact C^∞ functions.

Let $\eta = \eta(U)$ be any strictly convex function, and construct the function

$$q(U) = \int_0^U f'(s)\eta'(s)ds.$$

Note that η and q satisfy the relation

$$q' = \eta' f'. \tag{3.27}$$

Multiplying $\eta'(U)$ to both sides of (3.26) and using the chain rule and the relation (3.27), we obtain

$$\begin{aligned}
& \eta'(U^\varepsilon)U_t^\varepsilon + \eta'(U^\varepsilon)f'(U^\varepsilon)U_x^\varepsilon = \varepsilon\eta'(U^\varepsilon)U_{xx}^\varepsilon \\
\Rightarrow\quad & \eta'(U^\varepsilon)U_t^\varepsilon + q'(U^\varepsilon)U_x^\varepsilon = \varepsilon\eta'(U^\varepsilon)U_{xx}^\varepsilon \\
\Rightarrow\quad & \eta(U^\varepsilon)_t + q(U^\varepsilon)_x = \varepsilon\eta(U^\varepsilon)_{xx} - \varepsilon\eta''(U^\varepsilon)(U_x^\varepsilon)^2.
\end{aligned}$$

Since η is a convex function, we have $\eta'' \geqslant 0$. Therefore, we obtain

$$\eta(U^\varepsilon)_t + q(U^\varepsilon)_x \leqslant \varepsilon\eta(U^\varepsilon)_{xx}. \tag{3.28}$$

It may be shown with the theory of reaction-diffusion equations that the limit $U = \lim_{\varepsilon\to 0} U^\varepsilon$ exists and is a weak solution of the conservation law (3.4). By (3.28), U satisfies

$$\eta(U)_t + q(U)_x \leqslant 0. \tag{3.29}$$

The function η is called an *entropy* function and the corresponding function q is called an *entropy flux*. The pair (η, q) is called an *entropy pair*. The inequality (3.29) is referred to as the *entropy inequality*, and holds for every convex function η. In other words, every convex function is an entropy function for a scalar conservation law.

Since weak solutions are not necessarily differentiable, we need to interpret the entropy inequality (3.29) in a weak manner.

Definition 3.7. *A function $U(x,t)$ is an* entropy solution *of* (3.4) *if it satisfies the following conditions:*

(i) $U \in L^\infty(\mathbb{R} \times \mathbb{R}_+)$.

(ii) U is an weak solution of (3.4)*, i.e, it satisfies the identity* (3.15) *for all test functions $\varphi \in C^1(\mathbb{R} \times \mathbb{R}_+)$.*

(iii) For all test functions $\varphi \in C_c^1(\mathbb{R} \times (0,\infty))$ with $\varphi \geqslant 0$, U satisfies

$$\int_{\mathbb{R}\times\mathbb{R}_+} \eta(U)\varphi_t + q(U)\varphi_x \geqslant 0, \tag{3.30}$$

for all entropy pairs (η, q).

Remark 3.8. *Any convex function η serves as an entropy function for a scalar conservation law. Of particular importance are the so-called Kruzkhov entropies:*

$$\eta = \bar{\eta}(U,k) = |U-k| \tag{3.31}$$

for constants $k \in \mathbb{R}$. The corresponding entropy flux is

$$q = q(U,k) = \operatorname{sign}(U-k)(f(U)-f(k)). \tag{3.32}$$

It is straightforward to check that any smooth convex function $g(U)$ can be approximated by a linear combination of functions like $\bar{\eta}(U,k)$ on bounded intervals $U \in [a,b]$. More precisely, given $\delta > 0$, there exist $N \in \mathbb{N}$ and $\alpha_k, c_k \in \mathbb{R}$ with $1 \leqslant k \leqslant N$ such that

$$\Big| g(U) - \sum_{k=1}^{N} \alpha_k |U - c_k| \Big| \leqslant \delta \qquad \textit{for all } U \in [a,b].$$

Thus, (3.29) *holds if and only if*

$$|U-k|_t + \big(\operatorname{sign}(U-k)(f(U)-f(k))\big)_x \leqslant 0 \tag{3.33}$$

(in the weak sense) for all $k \in \mathbb{R}$.

It is straightforward to check that the entropy solution satisfies the Oleinik entropy condition (check [**1**] for a proof). The concept of entropy solutions is the correct notion of solutions to conservation laws. We will demonstrate the existence and uniqueness of entropy solutions in this section.

The entropy inequality (3.29) can be used to obtain estimates on solutions. To see this, integrate (3.29) over space and integrate by parts to obtain

$$\frac{d}{dt}\int_{\mathbb{R}} \eta(U) \leqslant 0 \quad \Rightarrow \quad \int_{\mathbb{R}} \eta(U(x,t))dx \leqslant \int_{\mathbb{R}} \eta(U_0(x))dx. \tag{3.34}$$

Since the function η may be any convex function, we can choose $\eta(U) = \frac{U^2}{2}$ and obtain a bound on the entropy solution in L^2. This estimate is a *nonlinear* analogue of the energy estimate (2.6) for the linear transport equation. Choosing η as

$$\eta(U) = \frac{|U|^p}{p}$$

will lead to estimates in L^p spaces for all $p \geqslant 1$. Hence, the entropy inequality serves to yield stability estimates.

A key estimate is the L^∞ estimate that bounds the maximum and minimum of the solutions of (3.26). The L^∞ estimate is a consequence of the standard maximum principle for parabolic equations. A formal proof proceeds as follows: Assume that the maximum of the solution U is attained at a point x_0 at time $t_0 > 0$. Clearly $U_t(x_0,t_0) = U_x(x_0,t_0) = 0$. Furthermore, as (x_0,t_0) is a maximum, we can assume that $U_{xx}(x_0,t_0) < 0$. Substituting all these relations in (3.26) leads to a contradiction. Hence, the maximum can only be attained at the initial data. A similar estimate holds at minima.

Another crucial stability estimate for (3.26) concerns the control of the derivative. To motivate this estimate, we let $\eta(U) = \bar{\eta}(U,0) = |U|$, the Kruzkhov entropy centered at 0. Formally we have

$$\eta'(U) = \operatorname{sign}(U), \qquad \eta''(U) = \delta(U),$$

where δ is the Dirac mass or delta function centered at 0. Multiplying (3.26) with $-\eta'(U^\varepsilon_x)_x$ and using the chain rule and integrating by parts, we obtain

$$
\begin{aligned}
&-\int_{\mathbb{R}} \eta'(U^\varepsilon_x)_x U^\varepsilon_t\,dx - \int_{\mathbb{R}} \eta'(U^\varepsilon_x)_x f'(U^\varepsilon)U^\varepsilon_x\,dx = -\varepsilon\int_{\mathbb{R}} \eta'(U^\varepsilon_x)_x U^\varepsilon_{xx}\,dx \\
&\int_{\mathbb{R}} \eta'(U^\varepsilon_x) U^\varepsilon_{xt}\,dx - \int_{\mathbb{R}} \eta'(U^\varepsilon_x)_x f'(U^\varepsilon)U^\varepsilon_x\,dx = -\varepsilon\int_{\mathbb{R}} \eta'(U^\varepsilon_x)_x U^\varepsilon_{xx}\,dx \qquad \text{(integration by parts)} \\
&\frac{d}{dt}\int_{\mathbb{R}} |U^\varepsilon_x|\,dx - \underbrace{\int_{\mathbb{R}} \eta''(U^\varepsilon_x) f'(U^\varepsilon_x)U^\varepsilon_x U^\varepsilon_{xx}\,dx}_{=0} = -\varepsilon\int_{\mathbb{R}} \eta''(U^\varepsilon_x)(U^\varepsilon_{xx})^2\,dx \quad \text{(chain rule)},
\end{aligned}
$$

which in the limit $\varepsilon \to 0$ gives us

$$\frac{d}{dt}\int_{\mathbb{R}} |U_x|\,dx \leqslant 0.$$

These computations are formal but can be made completely rigorous (see [**1**]).

Of great importance in the context of conservation laws is the concept of total variation. Let g be a function defined on an interval $[a,b]$. The *total variation* of g is defined as

$$\|g\|_{TV} = \sup_{\mathcal{P}} \sum_{j=1}^{N_{\mathcal{P}}-1} |g(x_{j+1}) - g(x_j)|, \tag{3.35}$$

where the supremum is taken over all partitions $\mathcal{P} = \{a = x_1 < x_2 < \cdots < x_{N_{\mathcal{P}}} = b\}$ of the interval $[a,b]$. It is straightforward to check that if g is differentiable, then

$$\|g\|_{TV} = \int_a^b |g_x|\,dx.$$

The total variation is only a semi-norm; indeed, the total variation of any constant function is zero. We turn it into a norm by defining

$$\|g\|_{BV} = \|g\|_{L^1} + \|g\|_{TV}. \tag{3.36}$$

We define the space of functions with bounded variation (BV) as

$$BV(\mathbb{R}) = \left\{ g \in L^1(\mathbb{R}) \;:\; \|g\|_{BV} < \infty \right\}. \tag{3.37}$$

The above estimate on $\int |U_x|dx$ is a BV estimate. It states that the solutions to a conservation law (3.4) are *Total Variation Diminishing* (TVD), that is,

$$\|U(\cdot,t)\|_{BV} \leqslant \|U_0\|_{BV} \qquad \text{for all } t > 0.$$

We are now in a position to state the main well-posedness results for scalar conservation laws:

Theorem 3.9. *If* $U_0 \in L^\infty(\mathbb{R}) \cap BV(\mathbb{R})$*, then* (3.4) *has a entropy solution* U *which satisfies the estimates*

$$\|U(\cdot,t)\|_{L^\infty} \leqslant \|U_0\|_{L^\infty} \tag{3.38}$$

and

$$\|U(\cdot,t)\|_{TV} \leqslant \|U_0\|_{TV}. \tag{3.39}$$

Furthermore, if U *and* V *are entropy solutions of* (3.4) *with initial* U_0 *and* V_0*, respectively, then*

$$\int_{\mathbb{R}} |U(x,t) - V(x,t)|dx \leqslant \int_{\mathbb{R}} |U_0(x) - V_0(x)|dx \qquad \text{for all } t > 0. \tag{3.40}$$

Hence, entropy solutions are unique.

The proof of this theorem is outside the scope of these notes (consult standard text books like [**1**]). We only sketch the main ideas of the proof. The existence is based on the viscous approximation (3.26). The entropy conditions yield estimates on the solution in L^p spaces as discussed before. Formal arguments for establishing the L^∞ and BV bounds are discussed above. Uniqueness of entropy solutions relying on the L^1 contraction estimate (3.40) is based on the ingenious *doubling of variables* idea of Kruzkhov [**5**] and uses the Kruzkhov entropies (3.31) extensively.

Summarizing the theoretical discussion of this section, we have the following results:

- Solutions of the conservation law (3.4) may develop discontinuities or shock waves, even for smooth initial data. Consequently, weak solutions are sought. Shock speeds are computed with the Rankine-Hugoniot condition (3.16).
- Weak solutions are not necessarily unique. Entropy conditions like the Oleinik entropy conditions have to be imposed. Self-similar continuous solutions or rarefaction waves have to be considered.
- Explicit solutions for the Riemann problem (even for non-convex fluxes) can be constructed in terms of shocks, rarefaction waves and compound shocks.
- Entropy solutions exist and are unique. Furthermore, the entropy solutions satisfy an L^∞ estimate, L^p estimates and are TVD – that is, the BV-norm decreases in time.

Despite the considerable theoretical results, we cannot obtain explicit solution formulas for more complicated initial data. Hence, we have to design efficient numerical methods for computing these solutions.