ETH Zürich FS 2015
D-MATH
Prof. Antti Knowles

Coordinator
Thomas Cayé

Applied Stochastic Processes

Exercise Sheet 3

Please hand in by 12:00 on Tuesday 17.03.2015 in the assistant's box in front of HG E 65.1

Exercise 3.1

Let $(X_i)_{i\in\mathbb{N}}$ be a sequence of square-integrable i.i.d. random variables with $\mathbb{E}[X_i] = \mu$, $\text{Var}[X_i] = \sigma^2$ and τ a non-negative integer-valued random variable independent of $(X_i)_{i\in\mathbb{N}}$. For $n \in \mathbb{N}_0$ define $S_n := \sum_{i=1}^n X_i$.

(a) Suppose that $\mathbb{E}[\tau] < \infty$. Show that

$$\mathbb{E}[S_\tau|\tau] = \mu\tau \text{ a.s.} \quad \text{and} \quad \mathbb{E}[S_\tau] = \mu\mathbb{E}[\tau].$$

Hint: Do not forget to argue that S_τ is integrable.

(b) Suppose that $\mathbb{E}[\tau^2] < \infty$. Show that

$$\mathbb{E}[(S_\tau)^2|\tau] = \sigma^2\tau + \mu^2\tau^2 \text{ a.s.} \quad \text{and} \quad \text{Var}[S_\tau] = \sigma^2\mathbb{E}[\tau] + \mu^2\text{Var}[\tau].$$

The above formulas are known as *Wald's equations*.

Exercise 3.2

Let $(N_t)_{t\geq 0}$ be a standard Poisson process with rate $\lambda > 0$ and $(X_k)_{k\in\mathbb{N}}$ a sequence of real-valued i.i.d. random variables with common distribution μ such that $(N_t)_{t\geq 0}$ and $(X_k)_{k\in\mathbb{N}}$ are independent. Define the process $Z = (Z_t)_{t\geq 0}$ by

$$Z_t := \sum_{k=1}^{N_t} X_k, \quad t \geq 0.$$

Z is called a *compound Poisson process* with rate λ and *jump size distribution* μ.

(a) For $t > 0$ determine the distribution and the characteristic function of Z_t.

(b) Prove that Z has stationary and independent increments.

(c) Show that if $P[X_i = 1] = 1 - P[X_i = 0] = p$, then Z is a Poisson process with rate λp.

Exercise 3.3

Theorem 3 of the lecture states that, for $\lambda > 0$, and $(N_t)_{t\geqslant 0}$ a counting process with $N_0 = 0$ and jumps of size 1 $\mathbb{P}$-a.s. the following statements, among others, are equivalent (we use the numbering of the lecture)

(ii) $(N_t)_{t \geqslant 0}$ has independent and stationary increments, and N_t is a $\mathrm{Poi}(\lambda t)$ random variable for all t.

(iii) The successive jump times $(S_i)_{i \geqslant 1}$ are $\mathbb{P}$-a.s. finite and $(T_i)_{i \geqslant 1}$ defined by $T_i := S_i - S_{i-1}$ are i.i.d. $\mathrm{Exp}(\lambda)$ random variables.

In the proof of the implication (ii) $\Rightarrow$ (iii) in Theorem 3 of the lecture the joint distribution of (S_1, S_2) was computed, and it was showed that S_1 and S_2 are $\mathbb{P}$-a.s. finite.

Extend the proof given in the lecture to obtain the distribution of the random vector $(S_1, S_2, \ldots, S_k)$ for any $k \in \mathbb{N}$, prove that S_i is finite for any $i \geqslant 1$, and prove the implication (ii) $\Rightarrow$ (iii) for any positive integer k.