ETH Zürich FS 2015
D-MATH
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Applied Stochastic Processes

Exercise Sheet 1

Please hand in by 12:00 on Tuesday 03.03.2015 in the assistant's box in front of HG E 65.1

Exercise 1.1

Let X be a random variable, and suppose that the moment generating function $f(t) := \mathbb{E}[e^{tX}]$ is finite for t in some open neighbourhood I of 0. Then f is smooth in I. (If you are not sure why, prove it!) Moreover, the n-th moment of X is simply $f^{(n)}(0)$.

We define the *cumulant generating function* F on I through $F(t) := \log f(t)$. The n-th cumulant of X is defined as

$$c_n := F^{(n)}(0).$$

(a) Prove that the first two cumulants of X are $c_1 = \mathbb{E}[X]$ and $c_2 = \mathrm{Var}[X]$.

(b) Let X be a Poisson-distributed random variable with parameter $\lambda > 0$. Prove that $c_n = \lambda$ for all $n \in \mathbb{N}$.

We conclude that all cumulants of a Poisson random variable coincide, and are in particular equal to its expectation.

Exercise 1.2

Let $T_1, \ldots, T_k$ be i.i.d. random variables with distribution $\mathrm{Exp}(\lambda)$. Show that $\sum_{i=1}^k T_i$ has distribution $\mathrm{Gamma}(k, \lambda)$.

Recall that the $\mathrm{Gamma}(\nu, \lambda)$-distribution has the density

$$f_{\Gamma(\nu,\lambda)}(t) = \lambda^\nu \frac{t^{\nu-1}}{\Gamma(\nu)} e^{-\lambda t}\, \mathbf{1}(t > 0)\,.$$

Here ν and λ are positive parameters (which need not be integer-valued).

Exercise 1.3

Suppose that countably many points are randomly distributed in $\mathbb{R}^2$, and denote by $N(A)$ the number of points in the Borel set $A \subset \mathbb{R}^2$. Suppose that $N(A)$ has distribution $\mathrm{Poisson}(\lambda|A|)$, and that for disjoint Borel sets $A_1, \ldots, A_k$ the random variables $N(A_1), \ldots, N(A_k)$ are independent. Here $\lambda > 0$ is a constant, and $|A|$ denotes the Lebesgue measure of A.

(a) For $r > 0$ set $B_r := \{x \in \mathbb{R}^2 : |x| < r\}$ and $D := \inf\{r > 0 : N(B_r) > 0\}$. Determine the distribution function and the density of D.

(b) Take radii $R > r > 0$ and define

$$f(R, r) := \mathbb{P}[N(B_R) = 1 \,|\, N(B_r) = 1].$$

Prove that

$$\lim_{R \downarrow 0} \lim_{r \downarrow 0} f(R, r) = \lim_{r \downarrow 0} \lim_{R \downarrow r} f(R, r),$$

and explain the intuition behind this result.