ETH Zürich FS 2015
D-MATH
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Applied Stochastic Processes

Exercise Sheet 5

Please hand in by 12:00 on Tuesday 31.03.2015 in the assistant's box in front of HG E 65.1

Exercise 5.1

Renewal process with delay. Let $S_0 \geqslant 0$ be a random variable independent of an i.i.d. sequence of random variables $(T_i)_{i\geqslant 1}$ valued in $[0,\infty)$ and such that $\mathbb{P}[T_i = 0] < 1$ for all $i \in \mathbb{N}$. We define the random variables

$$S_n := S_0 + T_1 + \ldots + T_n,\ n \in \mathbb{N}.$$

The process defined by

$$N_t := \sum_{k=1}^{\infty} \mathbf{1}\left(S_k \leqslant t\right) = \sup\left\{n \in \mathbb{N}_0 \mid S_n \leqslant t\right\}$$

is a renewal process with delay.

We denote by F the cumulative distribution function of the T_i's and by G the cumulative distribution function of S_0. We define the renewal function $M(t) := \mathbb{E}\left[N_t\right]$.

The aim of the exercise is to prove a result similar to the lemma 2 in the lecture notes for renewal processes with delay, as well as to prove that the renewal function of such a process satisfies the $(G * F, F)$-renewal equation.

(a) Show that for all $t \geqslant 0$, and $r \in \mathbb{N}$, we have $\mathbb{E}\left[N_t^r\right] < \infty$.

Hint: Use that the result holds for a renewal process without delay and that $S_0 \geqslant 0$ holds $\mathbb{P}$-a.s.

(b) Prove that M is nondecreasing and right-continuous and that it can be written as follows

$$M(t) = \sum_{k=1}^{\infty} G * F^{*k}(t).$$

(c) Show that the Laplace transform of M for $s \geqslant 0$ is given by

$$\hat{M}(s) := \int_0^{\infty} \mathrm{e}^{-sx} dM(x) = \hat{G}(s)\frac{\hat{F}(s)}{1-\hat{F}(s)},$$

where $\hat{F}$ and $\hat{G}$ are the Laplace transforms of F and G.

(d) Prove that M satisfy the $(G * F, F)$-renewal equation.

Exercise 5.2

Let $(N_t)_{t\geq 0}$ be a renewal process with interarrival times having the density

$$f(x) = \lambda^2 x \mathrm{e}^{-\lambda x} \mathbb{1}_{\{x\geq 0\}}, \quad \lambda > 0.$$

Compute the renewal function $M(t) := E[N_t]$ and its asymptotic growth rate $\lim_{t\to\infty} \frac{M(t)}{t}$.

Exercise 5.3

Let $(U_i, V_i)_{i\in\mathbb{N}}$ be a sequence of i.i.d. random vectors with $U_i \geq 0$, $V_i \geq 0$. Assume that $T_i = U_i + V_i$ is not almost surely equal to 0 and denote by F its distribution function. We interpret U_i and V_i as alternating periods when a given machine is operational or in repair. The period U_1 begins at time 0. For $t \geq 0$ we define $Y_t := 1$ if the machine is operational at time t and $Y_t := 0$ otherwise. Let $g(t) := P[Y_t = 1]$ denote the probability of the machine being operational at time t.

(a) Prove that g satisfies the renewal equation

$$g(t) = P[U_1 > t] + \int_0^t g(t-s)\, \mathrm{d}F(s), \quad t \geq 0.$$

(b) Suppose further that $U_i \sim \mathrm{Exp}(\lambda)$ and $V_i \sim \mathrm{Exp}(\mu)$ for $i \in \mathbb{N}$ and $\lambda, \mu > 0$. Show that in this case g satisfies

$$g(t) = h(t) + \int_0^t g(t-s) f(s)\, \mathrm{d}s, \quad t \geq 0,$$

where $h(t) = \mathrm{e}^{-\lambda t}\mathbb{1}_{\{t\geq 0\}}$ and $f(t) = \frac{\lambda\mu}{\mu-\lambda}(\mathrm{e}^{-\lambda t} - \mathrm{e}^{-\mu t})\mathbb{1}_{\{t\geq 0\}}$.