ETH Zürich FS 2015
D-MATH
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Coordinator
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Applied Stochastic Processes

Exercise Sheet 6

Please hand in by 12:00 on Tuesday 14.04.2015 in the assistant's box in front of HG E 65.1

Exercise 6.1

Vehicles of random lengths arrive at a gate. Let L_k denote the length of the kth vehicle. We assume that the random variables L_k are i.i.d. with $E[L_k] < \infty$. The first vehicle that arrives parks directly at the gate. The vehicles arriving afterwards queue behind, leaving a random distance to the vehicle parked in front of their own. We assume that these distances are independent and uniformly distributed on $[0, 1]$. For $x \geq 0$, let N_x denote the number of vehicles parked in distance at most x from the gate. Compute $\lim_{x\to\infty} N_x/x$.

Exercise 6.2

Show that a renewal process with renewal function $M(t) = ct$, $t \geq 0$ for some constant $c > 0$ is a Poisson process.

Hint: The Laplace transform determines the distribution.

Exercise 6.3

Let N be a renewal process with interarrival times $(T_k)_{k\in\mathbb{N}}$ and renewal times $(S_k)_{k\in\mathbb{N}}$, i.e. $S_k := \sum_{j=1}^{k} T_j$. Denote by $\mathbb{F}$ the filtration generated by $(T_k)_{k\in\mathbb{N}}$, i.e. $\mathcal{F}_k := \sigma(T_1, \ldots, T_k)$ for all k, and let τ be an $\mathbb{F}$-*stopping time* with $\tau < \infty$ $\mathbb{P}$-a.s. Define the σ-field $\mathcal{F}_\tau$ by

$$\mathcal{F}_\tau := \{A \in \mathcal{F} \mid A \cap \{\tau = k\} \in \mathcal{F}_k \ \forall k \in \mathbb{N}_0\}.$$

a) Show that the process $(\widetilde{S}_k)_{k\geq 0}$ defined by

$$\widetilde{S}_k = S_{\tau+k} - S_\tau, \quad k \geq 0,$$

is equal in distribution to $(S_k)_{k\in\mathbb{N}}$ and independent from $\mathcal{F}_\tau$.

b) Suppose that $T_1 > 0$ $\mathbb{P}$-a.s. Show that the process $N^{(\tau)}$ define by

$$N_t^{(\tau)} = N_{S_\tau+t} - N_{S_\tau}, \quad t \geq 0,$$

is again a renewal process, which is equal in distribution to N and independent of $\mathcal{F}_\tau$.

Hint: Use part a).

Exercise 6.4

Let N be a renewal process with renewal times $(S_k)_{k\in\mathbb{N}_0}$, where $S_0 := 0$, and interarrival distribution F having finite mean $\mu > 0$. Denote by A and E the age and the excess process of N, respectively, i.e. $A_t := t - S_{N_t}$ and $E_t := S_{N_t+1} - t$, $t \geq 0$. For $x, y \geq 0$ set $Z_{(x,y)}(t) := \mathbb{P}[A_t \geq x, E_t > y]$.

a) Show that $Z_{(0,y)}$ satisfies the renewal equation

$$Z_{(0,y)}(t) = 1 - F(t+y) + \int_0^\infty Z_{(0,y)}(t-s)\,\mathrm{d}F(s), \quad t \geq 0.$$

b) Show that for $t \geq x$ we have $Z_{(x,y)}(t) = Z_{(0,x+y)}(t-x)$.

c) Assume that F is non-arithmetic. Compute $G_\infty(x,y) := \lim_{t\to\infty} Z_{(x,y)}(t)$. Deduce that the random vector (A_t, E_t) converges in distribution to some random vector (A_∞, E_∞) as $t \to \infty$.

d) Determine all distributions F for which A_∞ and E_∞ are independent.