ETH Zürich FS 2015
D-MATH
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Coordinator
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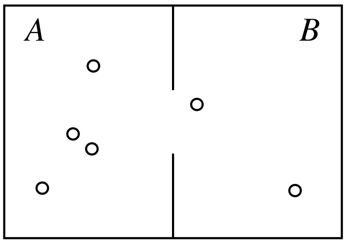
Applied Stochastic Processes

Exercise Sheet 9

Please hand in by 12:00 on Tuesday 05.05.2015 in the assistant's box in front of HG E 65.1

Exercise 9.1

Ehrenfest model. Consider N molecules moving between two compartments, as in the diagram below. Assume that at each step one of the molecules is chosen at random, and placed in the opposite compartment. Model the number of molecules in compartment A as a discrete-time Markov chain with state space $E = \{0, 1, \dots, N\}$.



(a) Write down the corresponding transition probabilities $r_{x,y}$.

(b) Find a reversible distribution π.

Exercise 9.2

Reversible Markov chains. Let $(X_n)_{n\in\mathbb{N}_0}$ be a Markov chain on a countable state space E, with transition probability $(r_{i,j})_{i,j\in E}$ and initial distribution μ. $(X_n)_{n\in\mathbb{N}_0}$ is called *reversible* if for all $m \in \mathbb{N}$ and all $i_0, \dots, i_m \in E$ we have

$$\mathbb{P}_\mu[X_0 = i_0, X_1 = i_1, \dots, X_m = i_m] = \mathbb{P}_\mu[X_m = i_0, X_{m-1} = i_1, \dots, X_0 = i_m].$$

a) Show that $(X_n)_{n\in\mathbb{N}_0}$ is reversible if and only if μ is a reversible distribution for $(X_n)_{n\in\mathbb{N}_0}$.

b) Let $F \subset E$ be a nonempty subset of E with $\sum_{j\in F} \mu_j > 0$. Let $(X'_n)_{n\in\mathbb{N}_0}$ be a Markov chain on F with transition probability $(r'_{i,j})_{i,j\in F}$ given by

$$r'_{ij} = \begin{cases} r_{ij} & \text{if } i \neq j, \\ r_{ii} + \sum_{k\in E\setminus F} r_{ik} & \text{if } i = j, \end{cases}$$

and initial distribution μ' given by

$$\mu'_i := \frac{\mu_i}{\sum_{j\in F} \mu_j}.$$

Prove that $(X'_n)_{n\in\mathbb{N}_0}$ is reversible if $(X_n)_{n\in\mathbb{N}_0}$ is.

Exercise 9.3

Show that all states of an irreducible Markov chain on a finite state space are positive recurrent.

Exercise 9.4

Recurrence and stationarity. Consider the Markov chain $(X_n)_{n\in\mathbb{N}_0}$ with state space $\mathbb{N}$, and transition probability

$$r_{ij} = \begin{cases} \pi(j) & \text{if } i = 1,\ j \geq 1, \\ 1 & \text{if } i > 1,\ j = i - 1, \\ 0 & \text{else,} \end{cases}$$

where π denotes a probability distribution on $\mathbb{N}$ with $\sum_{i\in\mathbb{N}} i\pi(i) < \infty$.

(a) Determine the positive recurrent, null recurrent, and transient states.

(b) Find a stationary distribution ν. Is this distribution reversible?

Exercise 9.5

Markov chains on graphs. Let $0 < p < 1$, $q = 1 - p$ and $0 < r < 1$. For the Markov chains on the graphs shown below, answer the following questions.

(i) Which states are recurrent and which are transient?

(ii) Determine the equivalence classes of the recurrent states.

(iii) Compute the n-step transition matrix $P^{(n)} = (P_{ij}^{(n)})_{ij}$.

(iv) Compute the limit $\lim_{n\to\infty} P^{(n)}$ (if it exists).