ETH Zürich FS 2015
D-MATH
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Applied Stochastic Processes

Solution Sheet 4

Solution 4.1

Let $t \geqslant 0$.

(a) For all $x \geqslant 0$:

$$\mathbb{P}[\gamma_t > x] = \mathbb{P}[S_{N_t+1} > t + x] = \mathbb{P}[N_{t+x} - N_t = 0] = \mathbb{P}[N_x = 0] = \exp(-\lambda x) \ .$$

Thus, the distribution function F_{γ_t} of γ_t is given by

$$F_{\gamma_t}(x) = \mathbf{1}(x \geqslant 0)(1 - \exp(-\lambda x)) .$$

For all $x \geqslant 0$:

$$\begin{aligned}\mathbb{P}[\delta_t \geqslant x] = \mathbb{P}[S_{N_t} \leq t - x] &= \mathbf{1}(0 \leq x \leq t)\,\mathbb{P}[N_t - N_{t-x} = 0] \\ &= \mathbf{1}(0 \leq x \leq t)\,\mathbb{P}[N_x = 0] = \mathbf{1}(0 \leq x \leq t)\exp(-\lambda x) \ .\end{aligned}$$

This implies

$$F_{\delta_t}(x) = \mathbf{1}(0 \leq x < t)(1 - \exp(-\lambda x)) + \mathbf{1}(x \geqslant t) .$$

(b) For all $x, y \geqslant 0$:

$$\begin{aligned}\mathbb{P}[\gamma_t > x, \delta_t \geqslant y] &= \mathbf{1}(0 \leq y \leq t)\,\mathbb{P}[S_{N_t+1} > t + x, S_{N_t} \leq t - y] \\ &= \mathbf{1}(0 \leq y \leq t)\,\mathbb{P}[N_{t+x} - N_{t-y} = 0] \\ &= \mathbb{P}[N_{t+x} - N_t = 0]\,\mathbf{1}(0 \leq y \leq t)\,\mathbb{P}[N_t - N_{t-y} = 0] \\ &= \mathbb{P}[\gamma_t > x]\,\mathbb{P}[\delta_t \geqslant y] \ .\end{aligned}$$

This implies that γ_t and δ_t are independent. Hence, $P[\gamma_t \leq x, \delta_t \leq y] = F_{\gamma_t}(x) F_{\delta_t}(y)$.

(c) The distribution function F_{β_t} is given as the convolution of F_{γ_t} and F_{δ_t}:

$$F_{\beta_t}(x) = \int_{\mathbb{R}} F_{\gamma_t}(x - y) dF_{\delta_t}(y) .$$

For $0 \leq x < t$:

$$F_{\beta_t}(x) = \int_0^x (1 - \mathrm{e}^{-\lambda(x-y)})\lambda \mathrm{e}^{-\lambda y} dy = 1 - \exp(-\lambda x)(1 + \lambda x) .$$

For $x \geqslant t$:

$$\begin{aligned}&F_{\beta_t}(x) \\ &= \int_0^t (1 - \exp(-\lambda(x - y)))\lambda \exp(-\lambda y)\, dy + (1 - \exp(-\lambda(x - t)))\exp(-\lambda t) \\ &= 1 - \exp(-\lambda x)(1 + \lambda t) .\end{aligned}$$

(d) F_{β_t} has the density

$$f_{\beta_t}(x) = \mathbf{1}(0 \leq x < t)\,\lambda^2 x \exp(-\lambda x) + \mathbf{1}(x \geqslant t)\,\lambda(1+\lambda t)\exp(-\lambda x)\,.$$

Hence,

$$\mathrm{E}[\beta_t] = \int_0^\infty x f_{\beta_t}(x)dx = \frac{2-\exp(-\lambda t)}{\lambda}\,,$$

It follows for $t > 0$

$$\mathrm{E}[\beta_t] > \frac{1}{\lambda} = \mathrm{E}[T_i]\,,$$

and

$$\lim_{t\to\infty} \mathrm{E}[\beta_t] = \frac{2}{\lambda} = 2\mathrm{E}[T_i]\,.$$

We discover that the interval in which t falls is not a "‘typical"’ interval. To give a short explanation note that the probability of $t > 0$ lying in a large interval is larger than the probability of t being contained in a short interval. See section 5.2 in *Queueing systems* by L. Kleinrock.

Solution 4.2

Let $R(t) = \int_0^t \alpha s ds = \frac{\alpha}{2}t^2$. Then we can make the following time change

$$\{W_n \leq t\} = \{N_t \geq n\} = \{\tilde{N}_{R(t)} \geq n\} = \{\tilde{W}_n \leq R(t)\},$$

where $(\tilde{N}_t)_{t\geq 0}$ is a homogeneous Poisson process with rate 1. We know that $\tilde{W}_n$ is Gamma$(n,1)$ distributed (as it is the sum of n i.i.d. Exp(1) random variables), hence

$$\begin{aligned}\mathbb{P}[W_n \leq t] &= \mathbb{P}[\tilde{W}_n \leq R(t)] \\ &= \int_0^{R(t)} \frac{1}{(n-1)!} s^{n-1} \mathrm{e}^{-s} ds \\ &= \int_0^t \frac{1}{(n-1)!} 2\left(\frac{\alpha}{2}\right)^n s^{2n-1} \mathrm{e}^{-\frac{\alpha}{2}s^2} ds.\end{aligned}$$

For $0 \leq s_1 \leq t_1 < s_2 \leq t_2$ we have

$$\begin{aligned}\mathbb{P}[s_1 < W_1 \leq t_1, s_2 < W_2 \leq t_2] &= \mathbb{P}[N_{s_1} = 0, N_{t_1} - N_{s_1} = 1, N_{s_2} - N_{t_1} = 0, N_{t_2} - N_{s_2} \geq 1] \\ &= \left(\mathrm{e}^{-R(s_2)} - \mathrm{e}^{-R(t_2)}\right)(R(t_1) - R(s_1)) \\ &= \int_{s_1 < y_1 \leq t_1, s_2 < y_2 \leq t_2} \mathrm{e}^{-R(y_2)} \rho(y_1)\rho(y_2) dy_1 dy_2,\end{aligned}$$

using that $(N_t)_{t\geq 0}$ has independent increments and the fact that $N_t - N_s$ for $0 \leq s < t$ is Poisson distributed with parameter $(R(t) - R(s))$.

The density of the joint distribution of (W_1, W_2) is given by

$$f(y_1, y_2) = \mathrm{e}^{-R(y_2)}\rho(y_1)\rho(y_2)\mathbf{1}(0 < y_1 < y_2) = \alpha^2 y_1 y_2 \mathrm{e}^{-\frac{\alpha}{2}y_2^2}\mathbf{1}(0 < y_1 < y_2)\,.$$

Let $h(t_1, t_2) = (t_1, t_1 + t_2)$. Then we obtain

$$\begin{aligned}\mathbb{P}[(W_1, W_2 - W_1) \in A] &= \mathbb{P}[(W_1, W_2) \in h(A)] \\ &= \int_{h(A)} f(y_1, y_2) dy_1 dy_2 \\ &= \int_A f \circ h(y_1, y_2) dy_1 dy_2 \\ &= \int_A \alpha^2 y_1 (y_1 + y_2) \mathrm{e}^{-\frac{\alpha}{2}(y_1+y_2)^2} \mathbf{1}(y_1 > 0)\,\mathbf{1}(y_2 > 0)\, dy_1 dy_2.\end{aligned}$$

Hence the density of $(W_1, W_2 - W_1)$ is given by

$$f_{(W_1,W_2-W_1)}(y_1,y_2) = \alpha^2 y_1(y_1+y_2)\mathrm{e}^{-\frac{\alpha}{2}(y_1+y_2)^2}\mathbf{1}(y_1>0)\,\mathbf{1}(y_2>0)$$
$$= \underbrace{\left(\alpha y_1 \mathrm{e}^{-\frac{\alpha}{2}y_1^2}\mathbf{1}(y_1>0)\right)}_{\text{density of } W_1}\underbrace{\left(\alpha(y_1+y_2)\mathrm{e}^{-\frac{\alpha}{2}y_2^2-\alpha y_1 y_2}\mathbf{1}(y_2>0)\right)}_{\text{conditional density of } W_2-W_1 \text{ given } W_1}.$$

In particular W_1 and $W_2 - W_1$ are not independent (contrary to the homogeneous case).

Solution 4.3

(a) First we will show that $\mathbb{P}$-a.s. there exists n_0 such that for all $n \geq n_0$ we have

$$T_n \leq \frac{(1+\varepsilon)}{\lambda}\log(n/\lambda).$$

Set $E_n := \{T_n > \frac{(1+\varepsilon)}{\lambda}\log(n/\lambda)\}$, then

$$\mathbb{P}[E_n] = \exp\left(-\lambda\frac{(1+\varepsilon)}{\lambda}\log(n/\lambda)\right) = \left(\frac{\lambda}{n}\right)^{1+\varepsilon},$$

hence $\sum_n \mathbb{P}[E_n] < \infty$ and therefore by Borel-Cantelli we obtain $\mathbb{P}[\limsup_{n\to\infty} E_n] = 0$. This means that for P-a.a. ω there is $n_0(\omega)$ such that for all $n \geq n_0(\omega)$ we have

$$\max_{n_0(\omega)\leq k\leq n} T_k(\omega) \leq \frac{(1+\varepsilon)}{\lambda}\max_{n_0(\omega)\leq k\leq n}\log(k/\lambda) = \frac{(1+\varepsilon)}{\lambda}\log(n/\lambda).$$

Furthermore we can choose $n_1(\omega) \geq n_0(\omega)$ such that

$$\max_{1\leq k\leq n_0(\omega)} T_k(\omega) \leq \frac{(1+\varepsilon)}{\lambda}\log(n_1(\omega)/\lambda),$$

because log is a monotone function increasing to infinity. Therefore $\mathbb{P}$-a.s. there is $n_1 \in \mathbb{N}$, such that for all $n \geq n_1$ we have

$$\max_{1\leq k\leq n} T_k(\omega) \leq \frac{(1+\varepsilon)}{\lambda}\log(n/\lambda).$$

(b) We have $\limsup_{t\to\infty}\frac{N_t+1}{t} = \limsup_{t\to\infty}\frac{N_t}{t}$ and

$$\limsup_{t\to\infty}\frac{N_t}{t} \leq \limsup_{t\to\infty}\frac{N_t}{S_{N_t}} = \limsup_{k\to\infty}\frac{k}{S_k} = \lambda,$$

where we used in the last step that by the strong law of large numbers we have $S_k/k \to \frac{1}{\lambda}$ almost surely as $k\to\infty$. This implies that $\mathbb{P}$-a.s. there is t_0 such that for all $t > t_0$ we have

$$\frac{N_t+1}{t} \leq (1+\varepsilon)\lambda.$$

(c) $\mathbb{P}$-a.s. for t large enough we have

$$L_t \leq \max_{1\leq k\leq N_t+1} T_k \leq \frac{(1+\varepsilon)}{\lambda}\log\left(\frac{N_t+1}{\lambda}\right) \leq \frac{(1+\varepsilon)}{\lambda}\log(t(1+\varepsilon)),$$

which yields $\limsup_{t\to\infty}\frac{L_t}{\log t} \leq \frac{(1+\varepsilon)}{\lambda}$. As $\varepsilon > 0$ was arbitrarily chosen this yields the claim.