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Applied Stochastic Processes

Solution Sheet 6

Solution 6.1

For $k \in \mathbb{N}$ let U_k denote the distance between the $(k+1)$st and the kth vehicle queueing at the gate. Then for all $x \geq 0$,

$$N_x = 1 + \sum_{k=1}^{\infty} \mathbb{1}_{\left\{\sum_{j=1}^{k} L_j + U_j \leq x\right\}}.$$

Hence, $(N_x - 1)_{x \geq 0}$ is a renewal process with interarrival times $T_k = L_k + U_k$. Note that $E[T_k] = E[L_k] + \frac{1}{2}$ and that by assumption $E[L_k] < \infty$. The strong law of large numbers for renewal processes implies

$$\lim_{x \to \infty} \frac{N_x}{x} = \frac{1}{\frac{1}{2} + \mathrm{E}[L_k]} \quad \text{a.s.}$$

Solution 6.2

Lemma 2 (iii) form class yields $\hat{M}(x) = \frac{\hat{F}(x)}{1-\hat{F}(x)}$ for every $t \geq 0$. Using

$$\hat{M}(x) = \int_0^{\infty} \mathrm{e}^{-xu} \, \mathrm{d}M(u) = c \int_0^{\infty} \mathrm{e}^{-xu} \, \mathrm{d}u = \frac{c}{x},$$

we derive

$$\hat{F}(x) = \frac{\hat{M}(x)}{1 + \hat{M}(x)} = \frac{1}{x/c + 1}.$$

On the other hand, the Laplace transform of an $\mathrm{Exp}(c)$ random variable is given by $\int_0^{\infty} \mathrm{e}^{-xu} c \mathrm{e}^{-cu} \, \mathrm{d}u = \frac{1}{x/c+1}$. Because the Laplace transform determines the distribution, the interarrival times are $\mathrm{Exp}(c)$ distributed. We have a counting process starting at 0, with jump of size 1 $\mathbb{P}$-a.s. The interarrival time are $\mathbb{P}$-a.s. finite, independent and exponentially distributed with same parameter. By the Poisson process characterisation theorem, the renewal process is a Poisson process with rate c.

Solution 6.3

a) Let $B_1, \ldots, B_k \in \mathcal{B}(\mathbb{R})$ and $A \in \mathcal{F}_\tau$. Then we have by independence of $(T_{n+k})_{k \in \mathbb{N}}$ and $\mathcal{F}_n$

for all $n \in \mathbb{N}$, the fact that the T_k are i.i.d. and the law of total probability

$$\begin{aligned}
\mathbb{P}[\widetilde{S}_1 \in B_1, \ldots, \widetilde{S}_k \in B_k, A] &= \mathbb{P}[T_{\tau+1} \in B_1, \ldots, \sum_{j=1}^{k} T_{\tau+j} \in B_k, A] \\
&= \sum_{n=0}^{\infty} \mathbb{P}[T_{\tau+1} \in B_1, \ldots, \sum_{j=1}^{k} T_{\tau+j} \in B_k, A \cap \{\tau = n\}] \\
&= \sum_{n=0}^{\infty} \mathbb{P}[T_{n+1} \in B_1, \ldots, \sum_{j=1}^{k} T_{n+j} \in B_k, A \cap \{\tau = n\}] \\
&= \sum_{n=0}^{\infty} \mathbb{P}[T_{n+1} \in B_1, \ldots, \sum_{j=1}^{k} T_{n+j} \in B_k] \times \mathbb{P}[A \cap \{\tau = n\}] \\
&= \sum_{n=0}^{\infty} \mathbb{P}[T_1 \in B_1, \ldots, \sum_{j=1}^{k} T_j \in B_k] \times \mathbb{P}[A \cap \{\tau = n\}] \\
&= \mathbb{P}[T_1 \in B_1, \ldots, \sum_{j=1}^{k} T_j \in B_k] \times \mathbb{P}[A] \\
&= \mathbb{P}[S_1 \in B_1, \ldots, S_k \in B_k] \times \mathbb{P}[A]. \qquad (1)
\end{aligned}$$

For $A = \Omega$ this yields that $(\widetilde{S}_k)_{k\in\mathbb{N}}$ is equal in distribution to $(\widetilde{S}_k)_{k\in\mathbb{N}}$. Using this, we get for $A \in \mathcal{F}_\tau$ again arbitrary

$$\begin{aligned}
\mathbb{P}[\widetilde{S}_1 \in B_1, \ldots, \widetilde{S}_k \in B_k, A] &= \mathbb{P}[S_1 \in B_1, \ldots, S_k \in B_k] \times \mathbb{P}[A] \\
&= \mathbb{P}[\widetilde{S}_1 \in B_1, \ldots, \widetilde{S}_k \in B_k] \times \mathbb{P}[A], \qquad (2)
\end{aligned}$$

which shows that $(\widetilde{S}_k)_{k\in\mathbb{N}}$ and $\mathcal{F}_\tau$ are independent.

b) Since $T_1 > 0$ $\mathbb{P}$-a.s., it follows that on a set of full probability, N is a counting process starting at 0 and increasing by jumps of size 1. In particular, we have

$$S_k = \inf\{t \geq 0 \mid N_t = k\} \;\; \mathbb{P}\text{-a.s.}, \quad k \in \mathbb{N}, \qquad (3)$$

and $N_{S_k} = k$ $\mathbb{P}$-a.s..

Using these properties, it follows immediately that on a set of full probability, $N^{(\tau)}$ is a counting process starting at 0 and increasing by jumps of size 1, too. Denote by $(S_k^{(\tau)})_{k\in\mathbb{N}}$ the sequence of successive jump times of $N^{(\tau)}$. Using the notation from part a), it follows immediately from the definition of $N^{(\tau)}$ and the above that $S_k^{(\tau)} = \sum_{j=1}^{k} T_{\tau+k} = \widetilde{S}_k$ $\mathbb{P}$-a.s. for all $k \in \mathbb{N}$. Set $S_0^{(\tau)} := 0$ and $T_k^{(\tau)} := S_k^{(\tau)} - S_{k-1}^{(\tau)}$, $k \in \mathbb{N}$. Then it follows from part a) that $(T_k^{(\tau)})_{k\in\mathbb{N}}$ is equal in distribution to $(T_k)_{k\in\mathbb{N}}$ and independent from $\mathcal{F}_\tau$. Since a renewal process is characterised by its interarrival times, $\widetilde{N}^{(\tau)}$ is independent from $\mathcal{F}_\tau$.

Solution 6.4

a) Set $e_y(t) := \mathbb{P}[E_t \leq y]$. Then we have $e_y(t) = 1 - Z_{(0,y)}(t)$. Moreover, it follows from the lecture that e_y satisfies the renewal equation

$$e_y(t) = F(t+y) - F(t) + \int_0^t e_y(t-s)\,\mathrm{d}F(s), \quad t \geq 0. \qquad (4)$$

Using this, we get

$$
\begin{aligned}
Z_{(0,y)}(t) = 1 - e_y(t) &= 1 + F(t) - F(t+y) + \int_0^t (1 - e_y(t-s))\,\mathrm{d}F(s) - F(t) \\
&= 1 - F(t+y) + \int_0^t Z_{(0,y)}(t-s)\,\mathrm{d}F(s), \quad t \geq 0. \qquad (5)
\end{aligned}
$$

b) We have for $x, y \geq 0$ and $t \geq x$

$$
\begin{aligned}
\{A_t \geq x, E_t > y\} &= \{S_{N_t} \leq t - x, S_{N_t+1} > t + y\} \\
&= \{N_t = N_{t-x}, S_{N_t+1} > t + y\} \\
&= \{N_t = N_{t-x}, S_{N_{t-x}+1} > t + y\} \\
&= \{S_{N_{t-x}+1} > (t-x) + x + y\} \\
&= \{E_{t-x} > x + y\}. \qquad (6)
\end{aligned}
$$

This together with the definition of $Z_{(x,y)}$ implies the claim.

c) For $x \geq 0$ define $h_x(t) := (1 - F(t+x))\mathbb{1}_{\{t \geq 0\}}$. Then $h \geq 0$ is decreasing and satisfies

$$
\begin{aligned}
\int_0^\infty h_x(t)\,\mathrm{d}t &= \int_0^\infty (1 - F(t+x))\,\mathrm{d}t \leq \int_0^\infty (1 - F(t))\,\mathrm{d}t \\
&= \int_0^\infty \mathbb{P}[S_1 > t]\,\mathrm{d}t = \mathbb{E}[S_1] = \mu < \infty. \qquad (7)
\end{aligned}
$$

Hence h_x is directly Riemann integrable as shown in the lecture.

Putting the results of part a) and b) together with Smith's key renewal theorem, we get

$$
\begin{aligned}
\lim_{t\to\infty} Z_{(x,y)}(t) &= \lim_{t\to\infty} Z_{(0,x+y)}(t-x) = \lim_{t\to\infty} Z_{(0,x+y)}(t) \\
&= \frac{1}{\mu}\int_0^\infty h_{x+y}(u)\,\mathrm{d}u = \frac{1}{\mu}\int_{x+y}^\infty (1 - F(u))\,\mathrm{d}u. \qquad (8)
\end{aligned}
$$

For $x, y \geq 0$ set $G_\infty(x,y) := \frac{1}{\mu}\int_{x+y}^\infty (1 - F(u))\,\mathrm{d}u$. Note that

$$
G_\infty(0,0) = \frac{1}{\mu}\int_0^\infty (1 - F(u))\,\mathrm{d}u = \frac{\mu}{\mu} = 1. \qquad (9)
$$

Moreover, for $x, y \in \mathbb{R}$ define the function $\overline{G}_\infty$ by

$$
\overline{G}_\infty(x,y) := \begin{cases} G_\infty(-x,-y) & \text{if } x, y \leq 0, \\ G_\infty(-x,0) & \text{if } x \leq 0, y > 0, \\ G_\infty(0,-y) & \text{if } x > 0, y \leq 0, \\ 1 & \text{if } x, y > 0. \end{cases} \qquad (10)
$$

It follows directly from the definition of G that $\overline{G}_\infty$ is $[0,1]$-valued and continuous (and a fortiori right-continuous). In addition, it is not difficult to check that for $(x_1,y_1),(x_2,y_2) \in \mathbb{R}^2$ with $x_1 < x_2$ and $y_1 < y_2$ we have

$$
\overline{G}_\infty(x_2,y_2) - \overline{G}_\infty(x_1,y_2) - \overline{G}_\infty(x_2,y_1) + \overline{G}_\infty(x_1,y_1) \geq 0. \qquad (11)
$$

Moreover, we have $\overline{G}_\infty(0,0) = 1$ and $\lim_{(x,y)\to(-\infty,-\infty)} \overline{G}_\infty(x,y) = 0$.

In conclusion, $\overline{G}_\infty$ is the distribution function of a two dimensional random vector supported on $(-\infty,0]^2$. Put differently, there exists a random vector (A_∞, E_∞) valued in $[0,\infty)^2$ such that $\overline{G}_\infty$ is the distribution function of $(-A_\infty, -E_\infty)$ and we have

$$
G_\infty(x,y) = \mathbb{P}[A_\infty \geq x, E_\infty \geq y], \quad x, y \geq 0. \qquad (12)
$$

For $t \geq 0$, denote by $\overline{G}_t$ the distribution function of $(-A_t, -E_t)$ and define the function G_t on $[0,\infty)^2$ by $G_t(x,y) := \mathbb{P}[A_t \geq x, E_t \geq y]$. Note that G_t and $\overline{G}_t$ have the same relationship as G_∞ and $\overline{G}_\infty$.

We proceed to show that (A_t, E_t) converges in distribution to (A_∞, E_∞) as $t \to \infty$. This is clearly equivalent to showing that $(-A_t, -E_t)$ converges in distribution to (A_∞, E_∞) as $t \to \infty$, which in turn by continuity of G_∞ is equivalent to showing that

$$\lim_{t\to\infty} \overline{G}_t(x,y) = \overline{G}_\infty(x,y) \quad \text{for all } (x,y) \in \mathbb{R}^2. \tag{13}$$

Using the relationship between G_t and $\overline{G}_t$ and G_∞ and $\overline{G}_\infty$, respectively, the latter is equivalent to establishing that

$$\lim_{t\to\infty} G_t(x,y) = G_\infty(x,y) \quad \text{for all } x,y \geq 0. \tag{14}$$

Observe that $Z_{x,y}(t) = G_t(x, y+)$ for all $t,x,y \geq 0$, where $G_t(x,y+) = \lim_{u\downarrow y} G_t(x,u)$. By (8) it follows that

$$\lim_{t\to\infty} G_t(x,y+) = G_\infty(x,y) \quad \text{for all } x,y \geq 0. \tag{15}$$

Using continuity of G, monotonicity of $G_t(x,y+)$ in y and $\lim_{u\uparrow y} G_t(x,u+) = G_t(x,y)$, it is an easy exercise in analysis to derive (14).

d) A_∞ and E_∞ are independent if and only if for all $x,y \geq 0$ we have

$$\mathbb{P}[A_\infty \geq x, E_\infty \geq y] = \mathbb{P}[A_\infty \geq x]\mathbb{P}[E_\infty \geq y]. \tag{16}$$

Define the function g by $g(z) = \frac{1}{\mu}\int_z^\infty (1 - F(u))\,\mathrm{d}u$, $z \geq 0$. By part c) it follows that for all $x,y \geq 0$ we have

$$\mathbb{P}[A_\infty \geq x, E_\infty \geq y] = G(x,y) = g(x+y). \tag{17}$$

In particular A_∞ and E_∞ are independent if and only if for all $x,y \geq 0$ we have

$$g(x+y) = g(x)g(y). \tag{18}$$

As g is continuous, this functional equation has the unique solution $g(z) = e^{\alpha z}$, where $\alpha < 0$, since $\lim_{z\to\infty} g(z) = 0$. Moreover, we know that

$$e^{az} = g(z) = \frac{1}{\mu}\int_z^\infty (1 - F(s))\,\mathrm{d}s, \quad z \geq 0 \tag{19}$$

Differentiating both sides yields

$$ae^{az} = -\frac{1}{\mu}(1 - F(z)), \quad z \geq 0. \tag{20}$$

Hence, we have $F(z) = 1 + \mu\alpha e^{\alpha z}$. Plugging in $z = 0$ shows that $\alpha \geq -1/\mu$. Hence there exists $\lambda \in (0, 1/\mu]$ such that

$$F(t) = (1 - \lambda\mu e^{-\lambda t})\mathbb{1}_{\{t\geq 0\}} = (1-\lambda\mu) \times \mathbb{1}_{\{t\geq 0\}} + \lambda\mu(1 - e^{-\lambda t})\mathbb{1}_{\{t \geq 0\}}, \quad t \geq 0. \tag{21}$$

In conclusion, A_∞ and E_∞ are independent if and only if F is the *mixture* of a Dirac-distribution at 0 and an exponential distribution with parameter $\lambda \in (0, 1/\mu]$ with weights $(1-\lambda\mu)$ and $\lambda\mu$, respectively.

Remark: For $\lambda = 1/\mu$, N is a Poisson process with parameter $1/\mu$.