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D-MATH
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Applied Stochastic Processes

Solution Sheet 8

Solution 8.1

Assume that $(X_n)_{n\in\mathbb{N}}$ is the *canonical* Markov chain on $E = \{A, B, C, D, E, F\}$ with transition matrix

$$R = \begin{pmatrix} 0 & p & q & r & 0 & 0 \\ q & 0 & p & 0 & r & 0 \\ p & q & 0 & 0 & 0 & r \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Since D, E and F are *absorbing states* of $(X_n)_{n\in\mathbb{N}_0}$, we have to calculate $\rho_{A,D}$, $\rho_{A,E}$ and $\rho_{A,F}$. Note that due to the symmetry of the graph, we have

$$\begin{aligned} \rho_{B,D} &= \rho_{A,F}, \\ \rho_{C,D} &= \rho_{A,E}. \end{aligned} \tag{1}$$

To calculate $\rho_{A,D}$, we consider the first step X_1. Observe that the chain either jumps to state D with probability r, or to state B with probability p, or state C with probability q. If $X_1 = D$ the chain stays at D, if $X_1 = B$, the probability that the chain ends up in state D is $\rho_{C,D}$, and if $X_1 = B$, the probability that the chain ends up in state D is $\rho_{B,D}$. Therefore we obtain the equation

$$\rho_{A,D} = r + p\rho_{B,D} + q\rho_{C,D}.$$

Formally, this can be proved using the simple Markov property (Proposition 3.3):

$$\begin{aligned} \rho_{A,D} &= \mathbb{P}_A\big[H_D < \infty\big] \\ &= \mathbb{P}_A\big[H_D < \infty, X_1 = D\big] + \mathbb{P}_A\big[H_D < \infty, X_1 = B\big] + \mathbb{P}_A\big[H_D < \infty, X_1 = C\big] \\ &= \mathbb{P}_A[X_1 = D] + \mathbb{P}_A\big[H_D \circ \theta_1 < \infty, X_1 = B\big] + \mathbb{P}_A\big[H_D \circ \theta_1 < \infty, X_1 = C\big] \\ &= r + \mathbb{E}_A\big[\mathbb{1}_{\{H_D\circ\theta_1<\infty\}}\mathbb{1}_{\{X_1=B\}}\big] + \mathbb{E}_A\big[\mathbb{1}_{\{H_D\circ\theta_1<\infty\}}\mathbb{1}_{\{X_1=C\}}\big] \\ &= r + \mathbb{E}_A\big[\mathbb{E}_A\big[\mathbb{1}_{\{H_D\circ\theta_1<\infty\}}\mathbb{1}_{\{X_1=B\}} \,|\, \mathcal{F}_1\big]\big] + \mathbb{E}_A\big[\mathbb{E}_A\big[\mathbb{1}_{\{H_D\circ\theta_1<\infty\}}\mathbb{1}_{\{X_1=C\}} \,|\, \mathcal{F}_1\big]\big] \\ &= r + \mathbb{E}_A\big[\mathbb{1}_{\{X_1=B\}}\mathbb{E}_A\big[\mathbb{1}_{\{H_D\circ\theta_1<\infty\}} \,|\, \mathcal{F}_1\big]\big] + \mathbb{E}_A\big[\mathbb{1}_{\{X_1=C\}}\mathbb{E}_A\big[\mathbb{1}_{\{H_D\circ\theta_1<\infty\}} \,|\, \mathcal{F}_1\big]\big] \\ &= r + \mathbb{E}_A\big[\mathbb{1}_{\{X_1=B\}}\mathbb{E}_{X_1}\big[\mathbb{1}_{\{H_D<\infty\}}\big]\big] + \mathbb{E}_A\big[\mathbb{1}_{\{X_1=C\}}\mathbb{E}_{X_1}\big[\mathbb{1}_{\{H_D<\infty\}}\big]\big] \\ &= r + \mathbb{E}_A\big[\mathbb{1}_{\{X_1=B\}}\mathbb{E}_B\big[\mathbb{1}_{\{H_D<\infty\}}\big]\big] + \mathbb{E}_A\big[\mathbb{1}_{\{X_1=C\}}\mathbb{E}_C\big[\mathbb{1}_{\{H_D<\infty\}}\big]\big] \\ &= r + \mathbb{E}_A\big[\mathbb{1}_{\{X_1=B\}}\big]\mathbb{E}_B\big[\mathbb{1}_{\{H_D<\infty\}}\big] + \mathbb{E}_A\big[\mathbb{1}_{\{X_1=C\}}\big]\mathbb{E}_C\big[\mathbb{1}_{\{H_D<\infty\}}\big] \\ &= r + p\rho_{B,D} + q\rho_{C,D}. \end{aligned}$$

Using (1) we get

$$\rho_{A,D} = r + p\rho_{A,F} + q\rho_{A,E} = 1 - p - q + p\rho_{A,F} + q\rho_{A,E} \tag{2}$$

In an analogous way, we get

$$\rho_{A,E} = p\rho_{A,D} + q\rho_{A,F}, \tag{3}$$

$$\rho_{A,F} = p\rho_{A,E} + q\rho_{A,D}. \tag{4}$$

Solving the system of the three linear equations (2) – (4), we obtain

$$\rho_{A,D} = \frac{1-pq}{1-pq+p+q+p^2+q^2},$$
$$\rho_{A,E} = \frac{p+q^2}{1-pq+p+q+p^2+q^2},$$
$$\rho_{A,F} = \frac{q+p^2}{1-pq+p+q+p^2+q^2}.$$

Solution 8.2

For $k=0$ the result is clear. If $y \in C$, then $P_y[\tau_C > kN] = P_y[0 > kN] = 0$ for all $k \geq 0$. We will prove the inequality for all $y \in E \backslash C$ and $k \geq 1$ by induction over k. For $y \in E\backslash C$, we have

$$P_y[\tau_C > N] \leq P_y[\tau_C > n(y)] \leq 1 - r_{y,C}(n(y)) \leq 1-\varepsilon. \tag{5}$$

For $k \geq 2$, we obtain

$$P_y[\tau_C > kN] = E_y\big[\mathbb{1}_{\{\tau_C > kN\}}\big] = E_y\big[E_y\big[\mathbb{1}_{\{\tau_C > kN\}} \mid \mathcal{F}_{(k-1)N}\big]\big].$$

Moreover,

$$\mathbb{1}_{\{\tau_C > kN\}} = \mathbb{1}_{\{\tau_C > (k-1)N\}}\big(\mathbb{1}_{\{\tau_C > N\}} \circ \theta_{(k-1)N}\big).$$

This can be seen by noting that $\{\tau_C > \ell\} = \{X_0, X_1, \ldots, X_\ell \in E\backslash C\}$ and therefore

$$\begin{aligned}\mathbb{1}_{\{\tau_C > N\}} \circ \theta_{(k-1)N} &= \mathbb{1}_{\{X_0,\ldots,X_N \in E\backslash C\}} \circ \theta_{(k-1)N} \\ &= \mathbb{1}_{\{X_{(k-1)N},\ldots,X_{kN} \in E\backslash C\}}.\end{aligned}$$

The function $\mathbb{1}_{\{\tau_C > (k-1)N\}} = \mathbb{1}_{\{X_0, X_1, \ldots, X_{(k-1)N} \in E\backslash C\}}$ is $\mathcal{F}_{(k-1)N}$-measurable. Applying the simple Markov property in the second step, we obtain

$$\begin{aligned}P_y[\tau_C > kN] &= E_y\big[\mathbb{1}_{\{\tau_C > (k-1)N\}} E_y\big[\mathbb{1}_{\{\tau_C > N\}} \circ \theta_{(k-1)N} \mid \mathcal{F}_{(k-1)N}\big]\big] \\ &= E_y\big[\mathbb{1}_{\{\tau_C > (k-1)N\}} \underbrace{E_{X_{(k-1)N}}\big[\mathbb{1}_{\{\tau_C > N\}}\big]}_{\leq (1-\varepsilon) \text{ by } (5)}\big] \\ &\leq (1-\varepsilon) \underbrace{E_y\big[\mathbb{1}_{\{\tau_C > (k-1)N\}}\big]}_{\leq (1-\varepsilon)^{k-1} \text{ by induction hypothesis}} \leq (1-\varepsilon)^k.\end{aligned}$$

Solution 8.3

(a) We have

$$h(x) = P_x[\tau_A < \tau_B] = E_x\big[\mathbb{1}_{\{\tau_A < \tau_B\}}\big] = E_x\big[E_x\big[\mathbb{1}_{\{\tau_A < \tau_B\}} \mid \mathcal{F}_1\big]\big]$$

and on $\{X_0 \in E\backslash(A\cup B)\}$ we obtain

$$\begin{aligned}
\mathbb{1}_{\{\tau_A<\tau_B\}} &= \sum_{n=1}^{\infty} \mathbb{1}_{\{\tau_A=n\}}\mathbb{1}_{\{\tau_B>n\}} \\
&= \sum_{n=1}^{\infty} \mathbb{1}_{\{X_1,\dots,X_{n-1}\in E\backslash A\}}\mathbb{1}_{\{X_n\in A\}}\mathbb{1}_{\{X_1,\dots,X_n\in E\backslash B\}} \\
&= \left(\sum_{n=1}^{\infty} \mathbb{1}_{\{X_0,\dots,X_{n-2}\in E\backslash A\}}\mathbb{1}_{\{X_{n-1}\in A\}}\mathbb{1}_{\{X_0,\dots,X_{n-1}\in E\backslash B\}}\right)\circ\theta_1 \\
&= \left(\sum_{n=0}^{\infty} \mathbb{1}_{\{X_0,\dots,X_{n-1}\in E\backslash A\}}\mathbb{1}_{\{X_n\in A\}}\mathbb{1}_{\{X_0,\dots,X_n\in E\backslash B\}}\right)\circ\theta_1 \\
&= \mathbb{1}_{\{\tau_A<\tau_B\}}\circ\theta_1.
\end{aligned}$$

Applying the simple Markov property for $x \in E\backslash(A\cup B)$ we obtain

$$h(x) = E_x\big[E_x\big[\mathbb{1}_{\{\tau_A<\tau_B\}}\circ\theta_1 \bigm| \mathcal{F}_1\big]\big] = E_x\big[E_{X_1}\big[\mathbb{1}_{\{\tau_A<\tau_B\}}\big]\big] = E_x[h(X_1)] = \sum_{y\in E} r_{x,y}h(y).$$

(b) We only need to show that

$$\forall x \in E\backslash(A\cup B) \quad \exists n(x) \text{ such that } r_{x,A\cup B}(n(x)) > 0, \tag{6}$$

then Exercise 8.2 implies

$$P_x[\tau_{A\cup B}=\infty] \le P_x[\tau_{A\cup B}>kN] \le (1-\varepsilon)^k \to 0 \text{ as } k\to\infty,$$

hence $P[\tau_{A\cup B}<\infty]=1$. We show (6) by contradiction. Suppose that there exists $x^* \in E\backslash(A\cup B)$ such that for all $n\ge 1$ we have $r_{x^*,A\cup B}(n)=0$. Then

$$P_{x^*}[\tau_{A\cup B}<\infty] = \sum_{n=1}^{\infty} \underbrace{P_{x^*}[\tau_{A\cup B}=n]}_{\le r_{x^*,A\cup B}(n)=0} = 0,$$

which contradicts the assumption of this exercise.

(c) We have

$$\begin{aligned}
E_\mu\big[h(X_{n\wedge\tau_{A\cup B}}) \bigm| \mathcal{F}_{n-1}\big] = E_\mu\big[&\mathbb{1}_{\{\tau_{A\cup B}<n\}}h(X_{n\wedge\tau_{A\cup B}}) \bigm| \mathcal{F}_{n-1}\big] \\
&+ E_\mu\big[\mathbb{1}_{\{\tau_{A\cup B}\ge n\}}h(X_{n\wedge\tau_{A\cup B}}) \bigm| \mathcal{F}_{n-1}\big].
\end{aligned}$$

On $\{\tau_{A\cup B}<n\}$ we have $X_{n\wedge\tau_{A\cup B}} = X_{(n-1)\wedge\tau_{A\cup B}}$, hence

$$E_\mu\big[\mathbb{1}_{\{\tau_{A\cup B}<n\}}h(X_{n\wedge\tau_{A\cup B}}) \bigm| \mathcal{F}_{n-1}\big] = \mathbb{1}_{\{\tau_{A\cup B}<n\}}h(X_{(n-1)\wedge\tau_{A\cup B}}), \tag{7}$$

as $\mathbb{1}_{\{\tau_{A\cup B}<n\}}$ is $\mathcal{F}_{n-1}$-measurable. The function $\mathbb{1}_{\{\tau_{A\cup B}\ge n\}}$ is also $\mathcal{F}_{n-1}$-measurable and on $\{\tau_{A\cup B}\ge n\}$ we have $X_{n\wedge\tau_{A\cup B}} = X_n$, therefore

$$\begin{aligned}
E_\mu\big[\mathbb{1}_{\{\tau_{A\cup B}\ge n\}}h(X_{n\wedge\tau_{A\cup B}}) \bigm| \mathcal{F}_{n-1}\big] &= \mathbb{1}_{\{\tau_{A\cup B}\ge n\}}E_\mu\big[h(X_n) \bigm| \mathcal{F}_{n-1}\big] \\
&= \mathbb{1}_{\{\tau_{A\cup B}\ge n\}}E_\mu\big[h(X_1)\circ\theta_{n-1} \bigm| \mathcal{F}_{n-1}\big] \\
&= \mathbb{1}_{\{\tau_{A\cup B}\ge n\}}E_{X_{n-1}}[h(X_1)],
\end{aligned}$$

using the simple Markov property in the last step. If $\tau_{A\cup B}\ge n$, then $X_{n-1}\in E\backslash(A\cup B)$. Using equation $(*)$ we obtain

$$E_{X_{n-1}}[h(X_1)] = \sum_{y\in E} r_{X_{n-1},y}h(y) = h(X_{n-1}),$$

hence

$$E_\mu\left[\mathbb{1}_{\{\tau_{A\cup B}\geq n\}} h(X_{n\wedge\tau_{A\cup B}}) \mid \mathcal{F}_{n-1}\right]$$
$$= \mathbb{1}_{\{\tau_{A\cup B}\geq n\}} h(X_{n-1}) = \mathbb{1}_{\{\tau_{A\cup B}\geq n\}} h(X_{(n-1)\wedge\tau_{A\cup B}}). \quad (8)$$

Combining (7) and (8) yields the claim.

In part a) of this exercise we showed that $h_1(x) = P[\tau_A < \tau_B]$ fulfills $(*)$. It is clear that h_1 is 1 on A and 0 on B. Let h_2 be a second solution of $(*)$ that is 1 on A and 0 on B. Then $h_1 - h_2$ also solves $(*)$ and is 0 on $A \cup B$. By b) we have $P[\tau_{A\cup B} < \infty] = 1$, so P-a.s. for n large we have

$$(h_1 - h_2)(X_{n\wedge\tau_{A\cup B}}) = 0,$$

hence $(h_1 - h_2)(X_{n\wedge\tau_{A\cup B}}) \to 0$ P-a.s. as $n \to \infty$. The function $h_1 - h_2$ is bounded as $E\backslash(A \cup B)$ is finite. Therefore also $E_\mu[(h_1 - h_2)(X_{n\wedge\tau_{A\cup B}})] \to 0$ as $n \to \infty$, which implies that $((h_1 - h_2)(X_{n\wedge\tau_{A\cup B}}))_{n\geq 0}$ is uniformly integrable. Martingale theory yields

$$(h_1 - h_2)(X_{n\wedge\tau_{A\cup B}}) = E_\mu[0 \mid \mathcal{F}_n] = 0$$

for all $n \geq 0$. This holds for all initial distributions μ, hence in particular for $\mu = \delta_x$, $x \in E\backslash(A \cup B)$. For $n = 0$, we obtain

$$(h_1 - h_2)(X_{0\wedge\tau_{A\cup B}}) = (h_1 - h_2)(x) = 0.$$

Solution 8.4

(a) One finds easily that for $x \in \mathbb{Z}$

$$(Re_\xi)(x) = \left(p\mathrm{e}^{\mathrm{i}\xi} + q\mathrm{e}^{-\mathrm{i}\xi}\right) e_\xi(x).$$

Let $n \geqslant 2$. By induction we get $(R^n e_\xi)(x) = \left(p\mathrm{e}^{\mathrm{i}\xi} + q\mathrm{e}^{-\mathrm{i}\xi}\right)^n e_\xi(x)$.

(b) We compute,

$$\begin{aligned}
\int_{[-\pi,\pi)} \frac{d\xi}{2\pi} (R^n e_\xi)(0) &= \int_{[-\pi,\pi)} \frac{d\xi}{2\pi} \sum_{y\in\mathbb{Z}} r_{0,y}(n) e_\xi(y) \\
&= \sum_{y\in\mathbb{Z}} r_{0,y}(n) \int_{[-\pi,\pi)} \frac{d\xi}{2\pi} e_\xi(y) \\
&= r_{0,0}(n),
\end{aligned}$$

where we used the dominated convergence for the second inequality as $|e_\xi| \leqslant 1$ and we proved in Exercise 7.3 that $||R(n)|| = 1$. Furthermore a quick computation yields $\int_{[-\pi,\pi)} \frac{d\xi}{2\pi} e_\xi(y) = \delta_{0,y}$, for $y \in \mathbb{Z}$.

Computing the same integral and using (a), we obtain

$$\begin{aligned}
\int_{[-\pi,\pi)} \frac{d\xi}{2\pi} (R^n e_\xi)(0) &= \int_{[-\pi,\pi)} \frac{d\xi}{2\pi} \left(p\mathrm{e}^{\mathrm{i}\xi} + q\mathrm{e}^{-\mathrm{i}\xi}\right)^n e_\xi(0) \\
&= \int_{[-\pi,\pi)} \frac{d\xi}{2\pi} \left(p\mathrm{e}^{\mathrm{i}\xi} + q\mathrm{e}^{-\mathrm{i}\xi}\right)^n.
\end{aligned}$$

This proves the claim.

(c) We have

$$\begin{aligned} K_\varepsilon &= \sum_{n\geqslant 0} \mathrm{e}^{-\varepsilon n} r_{0,0}(n) \\ &= \sum_{n\geqslant 0} \mathrm{e}^{-\varepsilon n} \int_{[-\pi,\pi)} \frac{d\xi}{2\pi} \left(p\mathrm{e}^{\mathrm{i}\xi} + q\mathrm{e}^{-\mathrm{i}\xi}\right)^n \\ &= \sum_{n\geqslant 0} \int_{[-\pi,\pi)} \frac{d\xi}{2\pi} \mathrm{e}^{-\varepsilon n} \left(p\mathrm{e}^{\mathrm{i}\xi} + q\mathrm{e}^{-\mathrm{i}\xi}\right)^n . \end{aligned}$$

We have $|\mathrm{e}^{-\varepsilon}\left(p\mathrm{e}^{\mathrm{i}x\xi} + q\mathrm{e}^{-\mathrm{i}x\xi}\right)| < 1$, so by the dominated convergence theorem we obtain

$$\begin{aligned} K_\varepsilon &= \int_{[-\pi,\pi)} \frac{d\xi}{2\pi} \sum_{n\geqslant 0} \mathrm{e}^{-\varepsilon n} \left(p\mathrm{e}^{\mathrm{i}\xi} + q\mathrm{e}^{-\mathrm{i}\xi}\right)^n \\ &= \int_{[-\pi,\pi)} \frac{d\xi}{2\pi} \frac{1}{1-\mathrm{e}^{-\varepsilon}\left(p\mathrm{e}^{\mathrm{i}\xi} + q\mathrm{e}^{-\mathrm{i}\xi}\right)}. \end{aligned}$$

By the monotone convergence theorem we have

$$\lim_{\varepsilon\to 0} K_\varepsilon = \sum_{n\geqslant 0} r_{0,0}(n).$$

So studying the limit of K_ε for ε going to 0 gives the behaviour of the random walk. If the limit is finite, the chain is transient, if not the chain is recurrent.
We have

$$\frac{1}{1-\mathrm{e}^{-\varepsilon}\left(p\mathrm{e}^{\mathrm{i}\xi} + q\mathrm{e}^{-\mathrm{i}\xi}\right)} = \frac{1}{1-\mathrm{e}^{-\varepsilon}\left(\cos(\xi) + \mathrm{i}a\sin(\xi)\right)}.$$

Let $\delta > 0$. For $\pi \geqslant |\xi| \geqslant \delta$, $\left|\frac{1}{1-\mathrm{e}^{-\varepsilon}(\cos(\xi)+\mathrm{i}a\sin(\xi))}\right| \leqslant \frac{1}{1-\mathrm{e}^{-\varepsilon}\cos(\delta)}$. Therefore,

$$\int_{\delta\leqslant|\xi|\leqslant\pi} \frac{d\xi}{2\pi} \frac{1}{1-\mathrm{e}^{-\varepsilon}\left(p\mathrm{e}^{\mathrm{i}\xi} + q\mathrm{e}^{-\mathrm{i}\xi}\right)} \leqslant \frac{\pi}{1-\mathrm{e}^{-\varepsilon}\cos(\delta)}.$$

- Assume $a = 0$, so that $p = q = \frac{1}{2}$. We have

  $$\int_{-\delta}^{\delta} \frac{d\xi}{2\pi} \frac{1}{1-\mathrm{e}^{-\varepsilon}\cos(\xi)} \to \infty,$$

  as ε goes to 0. We can indeed write

  $$\begin{aligned} 1-\mathrm{e}^{-\varepsilon}\cos(\xi) &= 1-(1+O(\varepsilon))\left(1-\frac{\xi^2}{2}+O(\xi^4)\right) \\ &= \frac{\xi^2}{2} + O(\varepsilon)\left(1-\frac{\xi^2}{2}+O(\xi^4)\right) + O(\xi^4) \end{aligned}$$

  which is not integrable around 0. So a symmetric random walk on $\mathbb{Z}$ is recurrent.
- Assume now $a > 0$. We need to study the quantity

  $$\int_{-\delta}^{\delta} \frac{d\xi}{2\pi} \frac{1}{1-\mathrm{e}^{-\varepsilon}\left(p\mathrm{e}^{\mathrm{i}\xi} + q\mathrm{e}^{-\mathrm{i}\xi}\right)} = \int_{-\delta}^{\delta} \frac{d\xi}{2\pi} \frac{1}{1-\mathrm{e}^{-\varepsilon}\left(\cos(\xi) + \mathrm{i}a\sin(\xi)\right)}.$$

We have

$$1-\mathrm{e}^{-\varepsilon}\left(\cos(\xi)+\mathrm{i}a\sin(\xi)\right) = 1-\left(1+\varepsilon+O(\varepsilon^2)\right)\left(1-\frac{\xi^2}{2}+\mathrm{i}a\xi+O(\xi^3)\right)$$
$$= -\mathrm{i}a\xi+O(\xi^2)-(\varepsilon+O(\varepsilon^2))\left(1+\mathrm{i}a\xi+O(\xi^2)\right),$$

and there exist two positive constants C_1 and C_2 such that for $|\xi|\leqslant\delta : C_1\varepsilon \leqslant \left|(\varepsilon+O(\varepsilon^2))\left(1+\mathrm{i}a\xi+O(\xi^2)\right)\right| \leqslant C_2\varepsilon$.
This gives

$$\left|\int_{-\delta}^{\delta}\frac{d\xi}{2\pi}\,\frac{1}{1-\mathrm{e}^{-\varepsilon}\left(\cos(\xi)+\mathrm{i}a\sin(\xi)\right)}\right| \leqslant \left|\int_{-\delta}^{\delta}\frac{d\xi}{2\pi}\frac{1}{-\mathrm{i}a\xi+C_1\varepsilon}\right|$$
$$= \frac{1}{|a|}\left|\int_{-\delta}^{\delta}\frac{d\xi}{2\pi}\frac{1}{\xi+\mathrm{i}\frac{C_1\varepsilon}{a}}\right|$$
$$= \frac{1}{|a|}\left|\int_{-\delta}^{\delta}\frac{d\xi}{2\pi}\frac{\xi}{\xi^2+\frac{C_1^2\varepsilon^2}{a^2}} - \mathrm{i}\int_{-\delta}^{\delta}\frac{d\xi}{2\pi}\frac{\frac{C_1\varepsilon}{a}}{\xi^2+\frac{C_1^2\varepsilon^2}{a^2}}\right|$$
$$= \frac{1}{|a|}\left|\frac{1}{2}\log\left(\frac{\delta^2+\frac{C_1^2\varepsilon^2}{a^2}}{\delta^2+\frac{C_1^2\varepsilon^2}{a^2}}\right) - \mathrm{i}\left(\mathrm{Arctan}\left(\frac{\delta a}{C_1\epsilon}\right)-\mathrm{Arctan}\left(-\frac{\delta a}{C_1\epsilon}\right)\right)\right|$$
$$= \frac{1}{|a|}\left|\left(\mathrm{Arctan}\left(\frac{\delta a}{C_1\varepsilon}\right)-\mathrm{Arctan}\left(-\frac{\delta a}{C_1\varepsilon}\right)\right)\right| \xrightarrow[\varepsilon\to 0]{} \frac{\pi}{|a|}.$$

The asymmetric random walk on $\mathbb{Z}$ is then transient.

(d) We define for $\xi\in[-\pi,\pi)^d$, the function $e_\xi(x)$ on $\mathbb{Z}^d$ by $e_\xi(x) := \mathrm{e}^{\mathrm{i}\cdot\xi}$. By a similar reasoning as above, we have

$$\left(Rf\right)(x) = \sum_{i=1}^{d}\left(\frac{b_i}{2}e_\xi(x+e_i)+\frac{b_i}{2}e_\xi(x-e_i)\right)$$

and

$$\int_{[-\pi,\pi)}\frac{d\xi}{(2\pi)^d}\left(R^n e_\xi\right)(0) = r_{0,0}(n)$$
$$= \int_{[-\pi,\pi)}\frac{d\xi}{2\pi}\left(\sum_{j=1}^{d}\frac{b_j}{2}\mathrm{e}^{\mathrm{i}\xi_j}+\frac{b_j}{2}\mathrm{e}^{-\mathrm{i}\xi_j}\right)^n$$
$$= \int_{[-\pi,\pi)}\frac{d\xi}{2\pi}\left(\sum_{j=1}^{d}\frac{b_j}{2}\cos(\xi_j)\right)^n.$$

We now define

$$K_\varepsilon^d = \sum_{n\geqslant 0}\mathrm{e}^{-\varepsilon n}r_{0,0}(n)$$
$$= \sum_{n\geqslant 0}\int_{[-\pi,\pi)}\frac{d\xi}{2\pi}\mathrm{e}^{-\varepsilon n}\left(\sum_{j=1}^{d}\frac{b_j}{2}\cos(\xi_j)\right)^n$$
$$= \int_{[-\pi,\pi)}\frac{d\xi}{2\pi}\frac{1}{1-\mathrm{e}^{-\varepsilon}\left(\sum_{j=1}^{d}\frac{b_j}{2}\cos(\xi_j)\right)}.$$

The last equality is obtained with the dominated convergence theorem. By monotone convergence we have $\lim_{\varepsilon\to 0}K_\varepsilon^d = \sum_{n\geqslant 0}r_{0,0}(n)$.

- Let us consider the case with $d = 2$. Let $\delta > 0$. Using a Taylor expansion of cosine, we get that for $\xi \in [-\delta, \delta]$, and δ small enough

$$\sum_{j=1}^{d} \frac{b_j}{2} \cos(\xi_j) \geqslant 1 - (\max_{j\in\{1,\dots,d\}} b_i + \epsilon_\delta)\frac{|\xi|^2}{2}$$

for some $\epsilon_\delta > 0$. Let $C_1 = \max_{j\in\{1,\dots,d\}} b_i + \epsilon_\delta$. Then for some $C_2, C_3 > 0$,

$$\begin{aligned}\int_{[-\delta,\delta)} \frac{d\xi}{2\pi} \frac{1}{1 - \mathrm{e}^{-\varepsilon}\left(\sum_{j=1}^{d} \frac{b_j}{2}\cos(\xi_j)\right)} &\geqslant \int_{[-\delta,\delta)} \frac{d\xi}{2\pi} \frac{1}{1 - \mathrm{e}^{-\varepsilon}(1 - C_1|\xi|^2)} \\ &\geqslant \int_{[-\delta,\delta)} \frac{d\xi}{2\pi} \frac{1}{C_2\varepsilon + C_3|\xi|^2} \\ &= \int_0^\delta \frac{r dr}{C_2\varepsilon + C_3 r^2} \\ &= \frac{1}{C_3}\log\left(\frac{C_2\varepsilon + \delta^2}{C_2\varepsilon}\right) \xrightarrow[\varepsilon\to 0]{} \infty\end{aligned}$$

We conclude that the symmetric random walk in $\mathbb{Z}^2$ is recurrent.

- Assume now $d \geqslant 3$. Similarly,

$$\sum_{j=1}^{d} \frac{b_j}{2} \cos(\xi_j) \leqslant 1 - (\min_{j\in\{1,\dots,d\}} b_i + \tilde{\epsilon_\delta})\frac{|\xi|^2}{2}$$

for some $\tilde{\epsilon_\delta} > 0$. Let $C_4 = \max_{j\in\{1,\dots,d\}} b_i - \tilde{\epsilon_\delta}$. Then, for some positive constants C_5, C_6, C_7,

$$\begin{aligned}\int_{[-\delta,\delta)} \frac{d\xi}{2\pi} \frac{1}{1 - \mathrm{e}^{-\varepsilon}\left(\sum_{j=1}^{d} \frac{b_j}{2}\cos(\xi_j)\right)} &\leqslant \int_{[-\delta,\delta)} \frac{d\xi}{2\pi} \frac{1}{1 - \mathrm{e}^{-\varepsilon}(1 - C_4|\xi|^2)} \\ &\leqslant \int_{[-\delta,\delta)} \frac{d\xi}{2\pi} \frac{1}{C_5\varepsilon + C_6|\xi|^2} \\ &= \int_0^\delta \frac{C_7 r^{d-1} dr}{C_5\varepsilon + C_6 r^2}\end{aligned}$$

which is finite for all $\varepsilon \geqslant 0$.
The random walk in $\mathbb{Z}^d$ for $d \geqslant 3$ is transient.

(e) A similar reasoning gives that asymmetric random walks in any dimension are transient.

Note: Let f be a function on $\mathbb{Z}^d$ with sufficiently fast decay (e.g. with compact support). For $\xi \in \big[-\pi, \pi\big)^d$ the Fourier transform of f is given by

$$\widehat{f}(\xi) := \sum_{x\in\mathbb{Z}^d} f(x)\mathrm{e}^{-\mathrm{i}\xi\cdot x}.$$

One can get f back using the transformation

$$f(x) = \int_{[-\pi,\pi)^d} \frac{d\xi}{(2\pi)^d} \widehat{f}(\xi)\mathrm{e}^{-\mathrm{i}\xi\cdot x}.$$

Let us define the scalar product for f and g square integrable by

$$\langle f, g\rangle := \sum_{x\in\mathbb{Z}^d} \overline{f(x)} g(x).$$

By Plancherel's theorem,

$$\langle f, g\rangle = \langle \widehat{f}, \widehat{g}\rangle := \int_{[-\pi,\pi)^d} \frac{d\xi}{(2\pi)^d} \overline{\widehat{f}}(\xi)\widehat{g}(\xi),$$

so that the Fourier transform is an isometry.

For a general random walk on $\mathbb{Z}^d$ as defined in question (e), define

$$\phi(\xi) = \sum_{j=1}^{d} \left(p_j e^{i\xi_j} + q_j e^{-i\xi_j}\right).$$

The Fourier transform operator diagonalises the operator R, by which we mean,

$$\widehat{(Rf)}(\xi) = \phi(\xi)\widehat{f}(\xi).$$

Hence,

$$\widehat{(R^n f)}(\xi) = \phi(\xi)^n \widehat{f}(\xi).$$

By definition,

$$r_{0,0}(n) = \langle \delta_0, R^n \delta_0\rangle.$$

Then,

$$\begin{aligned} r_{0,0}(n) &= \langle \widehat{\delta_0}, \widehat{R^n\delta_0}\rangle \\ &= \langle \widehat{\delta_0}, \phi^n \widehat{\delta_0}\rangle \\ &= \int_{[-\pi,\pi)^d} \frac{d\xi}{(2\pi)^d} \phi(\xi)^n, \end{aligned}$$

as $\widehat{\delta_0} = 1$.

To obtain the conditions for recurrence and transience of random walks we have actually used the fact that the Fourier transform diagonalises the operator R.