ETH Zürich FS 2015
D-MATH
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Applied Stochastic Processes

Solution Sheet 10

Solution 10.1

First note that all states in E are communicating. Therefore all states are either transient or recurrent. In addition, if all states are recurrent they are all either positive recurrent or null recurrent.

a) It suffices to establish that the state 0 is transient, i.e. we have to show that

$$\mathbb{P}_0[\widetilde{H}_0 < \infty] < 1. \tag{1}$$

After possibly augmenting the original probability space, let $(Y_n)_{n\in\mathbb{N}}$ be a sequence of i.i.d. random variables independent of $(X_n)_{n\in\mathbb{N}_0}$ under $\mathbb{P}_0$ with

$$\mathbb{P}_0[Y_i = 1] = p = 1 - \mathbb{P}_0[Y_i = -1]. \tag{2}$$

Set $S_n = \sum_{i=1}^n Y_i$ for $n \in \mathbb{N}_0$. Define $\widetilde{J}_{-1} := \inf\{n \geq 1; S_n = -1\}$. Noting that $X_1 = 1$ $\mathbb{P}_0$-a.s. we have

$$\mathbb{P}_0[\widetilde{H}_0 < \infty] = \mathbb{P}_1[\widetilde{H}_0 < \infty]. \tag{3}$$

Using that "$(X_n - X_{n-1})$ on $\mathbb{N}$ and Y_n on $\mathbb{N}_0$ have the same probabilistic structure (i.e. the same distribution)" it follows that

$$\mathbb{P}_1[\widetilde{H}_0 < \infty] = \mathbb{P}_0[\widetilde{J}_{-1} < \infty]. \tag{4}$$

Hence, it suffices to show that $\mathbb{P}_0[\widetilde{J}_{-1} < \infty] < 1$. This can be done in two different ways.

Markovian approach: Note that $(S_n)_{n\geq 0}$ is a Markov chain on $\mathbb{Z}$ starting at 0, with transition matrix given by

$$0 \to \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & & & & \\ \ddots & q & 0 & p & 0 & & & \\ & 0 & q & 0 & p & 0 & & \\ & & 0 & q & 0 & p & \ddots & \\ & & & \ddots & \ddots & \ddots & \ddots & \end{pmatrix} = R. \tag{5}$$

Seeking a contradiction suppose that $\mathbb{P}_0[\widetilde{J}_{-1} < \infty] = 1$, i.e. $\widetilde{\rho}_{0,-1} = 1$. By the special form of the transition matrix and since $p > q$ it follows that $\widetilde{\rho}_{-1,0} \geq \widetilde{\rho}_{0,-1} = 1$. But this implies that

$$\widetilde{\rho}_{0,0} \geq \widetilde{\rho}_{0,-1}\widetilde{\rho}_{-1,0} = 1, \tag{6}$$

i.e. the state 0 is recurrent for $(S_n)_{n\in\mathbb{N}_0}$. But this is a contradiction to the fact that $\lim_{n\to\infty} S_n = +\infty$ $\mathbb{P}_0$-a.s., which follows immediately from the strong law of large numbers.

Martingale approach: Denote by $\mathbb{G} := (\mathcal{G}_n)_{n\in\mathbb{N}_0}$ the filtration generated by $(S_n)_{n\in\mathbb{N}_0}$, i.e. $\mathcal{G}_n = \sigma(S_1,\dots,S_n) = \sigma(Y_1,\dots,Y_n)$, $n\in\mathbb{N}_0$. Note that Y_{n+1} and $\mathcal{G}_n$ are independent for all $n\in\mathbb{N}_0$. Moreover, $\widetilde{J}_{-1}$ is a $\mathbb{G}$-stopping time. Define the process $M=(M_n)_{n\geq 0}$ by

$$M_n := \left(\frac{q}{p}\right)^{S_n}. \tag{7}$$

Then M is a $\mathbb{G}$-martingale under $\mathbb{P}_0$ since for $n\in\mathbb{N}_0$ by independence of Y_{n+1} and $\mathcal{G}_n$ we have

$$\begin{aligned}\mathbb{E}_0[M_{n+1}\mid\mathcal{G}_n] &= \big[\mathbb{E}_0[M_n\times(q/p)^{Y_{n+1}}\mid\mathcal{G}_n\big] = M_n\mathbb{E}_0[(q/p)^{Y_{n+1}}]\\ &= M_n\times\left(p\times\frac{q}{p}+q\times\frac{p}{q}\right) = M_n\ \ \mathbb{P}_0\text{-a.s.}\end{aligned} \tag{8}$$

By the *optional stopping theorem*, we have

$$\mathbb{E}_0[M_{\widetilde{J}_{-1}\wedge\ell}] = \mathbb{E}_0[M_0] = 1 \quad \text{for all } \ell\in\mathbb{N}. \tag{9}$$

Since $q/p<1$ for $p>1/2$, we have $M_{\widetilde{J}_{-1}\wedge\ell}\leq(q/p)^{-1}=p/q$ for all $\ell\in N$. Hence by dominated convergence we have

$$\mathbb{E}_0[M_{\widetilde{J}_{-1}}\mathbb{1}_{\{\widetilde{J}_{-1}<\infty\}}] = \lim_{\ell\to\infty}\mathbb{E}_0[M_{\widetilde{J}_{-1}\wedge\ell}] = 1. \tag{10}$$

On the other hand we have

$$\mathbb{E}_0[M_{\widetilde{J}_{-1}}\mathbb{1}_{\{\widetilde{J}_{-1}<\infty\}}] = \mathbb{E}_0[p/q\mathbb{1}_{\{\widetilde{J}_{-1}<\infty\}}] = p/q\mathbb{P}_0[\widetilde{J}_{-1}<\infty]. \tag{11}$$

This implies that $\mathbb{P}_0[\widetilde{J}_{-1}<\infty]=q/p<1$.

b) Assume that $(X_n)_{n\in\mathbb{N}}$ is the canonical Markov chain with state space E and transition matrix R. First note that $\alpha=\rho_{1,0}=\rho_{2,1}$ by the special form of R. Using the normal and the strong Markov property we get

$$\begin{aligned}\alpha &= \rho_{1,0} = \mathbb{P}_1[\widetilde{H}_0<\infty] = \mathbb{P}_1[\widetilde{H}_0<\infty, X_1=0]+\mathbb{P}_1[\widetilde{H}_0<\infty, X_1=2]\\ &= \mathbb{P}_1[X_1=0]+\mathbb{P}_1[\widetilde{H}_0\circ\theta_1<\infty, X_1=2]\\ &= q+\mathbb{E}_1[\mathbb{E}_1[\mathbb{1}_{\{\widetilde{H}_0\circ\theta_1<\infty\}}\mathbb{1}_{\{X_1=2\}}\mid\mathcal{F}_1]]\\ &= q+\mathbb{E}_1[\mathbb{1}_{\{X_1=2\}}\mathbb{E}_{X_1}[\mathbb{1}_{\{\widetilde{H}_0<\infty\}}]\\ &= q+p\mathbb{E}_2[\mathbb{1}_{\{\widetilde{H}_0<\infty\}}] = q+p\mathbb{E}_2[\mathbb{1}_{\{\widetilde{H}_0<\infty\}}\mathbb{1}_{\{\widetilde{H}_1<\infty\}}]\\ &= q+p\mathbb{E}_2[\mathbb{1}_{\{\widetilde{H}_0\circ\theta_{\widetilde{H}_1}<\infty\}}\mathbb{1}_{\{\widetilde{H}_1<\infty\}}]\\ &= q+p\mathbb{E}_2[\mathbb{E}_2[\mathbb{1}_{\{\widetilde{H}_0\circ\theta_{\widetilde{H}_1}<\infty\}}\mathbb{1}_{\{\widetilde{H}_1<\infty\}}\mid\mathcal{F}_{\widetilde{H}_1}]]\\ &= q+p\mathbb{E}_2[\mathbb{1}_{\{\widetilde{H}_1<\infty\}}\mathbb{E}_1[\mathbb{1}_{\{\widetilde{H}_0<\infty\}}]]\\ &= q+p\mathbb{E}_2[\mathbb{1}_{\{\widetilde{H}_1<\infty\}}]\mathbb{E}_1[\mathbb{1}_{\{\widetilde{H}_0<\infty\}}]\\ &= q+p\alpha^2.\end{aligned} \tag{12}$$

The above equation has the solutions $\alpha=1$ and $\alpha=q/p$. Since, $\alpha\leq 1$ and $q/p\geq 1$ for $p\leq 1/2$, it follows immediately that $\alpha=1$ for $p\leq 1/2$. Finally, since $\rho_{0,1}=1$, we have for $q\leq 1/2$

$$\rho_{0,0}\geq\rho_{0,1}\rho_{1,0}=1, \tag{13}$$

i.e. 0 is a recurrent state.

(c) Again assume that $(X_n)_{n\in\mathbb{N}}$ is the canonical Markov chain with state space E and transition matrix R. First note that $\beta = \mathbb{E}_1[\widetilde{H}_0] = \mathbb{E}_2[\widetilde{H}_1]$ by the special form of R. Assuming that $(X_n)_{n\geq 0}$ is positive recurrent and using the normal and the strong Markov property we get

$$\begin{aligned}
\beta &= \mathbb{E}_1[\widetilde{H}_0] = \mathbb{E}_1[\widetilde{H}_0 \mathbb{1}_{\{X_1=0\}}] + \mathbb{E}_1[\widetilde{H}_0 \mathbb{1}_{\{X_1=1\}}] \\
&= \mathbb{E}_1[\mathbb{1}_{\{X_1=0\}}] + \mathbb{E}_1[\widetilde{H}_0 \mathbb{1}_{\{X_1=2\}}] \\
&= q + \mathbb{E}_1[(\widetilde{H}_0 \circ \theta_1 + 1)\mathbb{1}_{\{X_1=2\}}] \\
&= q + \mathbb{E}_1[\mathbb{E}_1[(\widetilde{H}_0 \circ \theta_1 + 1)\mathbb{1}_{\{X_1=2\}} \,|\, \mathcal{F}_1]] \\
&= q + \mathbb{E}_1[\mathbb{1}_{\{X_1=2\}} \mathbb{E}_{X_1}[\widetilde{H}_0 + 1]] \\
&= q + p\mathbb{E}_2[\widetilde{H}_0 + 1] = 1 + p\mathbb{E}_2[\widetilde{H}_0] \\
&= 1 + p\mathbb{E}_2[\widetilde{H}_0 \circ \theta_{\widetilde{H}_1} + \widetilde{H}_1] \\
&= 1 + p\mathbb{E}_2[\widetilde{H}_1] + p\mathbb{E}_2[\mathbb{E}_2[\widetilde{H}_0 \circ \theta_{\widetilde{H}_1} \,|\, \mathcal{F}_{\widetilde{H}_1}]] \\
&= 1 + p\beta + p\mathbb{E}_2[\mathbb{E}_1[\widetilde{H}_0]] = 1 + p\beta + p\mathbb{E}_1[\widetilde{H}_0] \\
&= 1 + 2p\beta. \qquad (14)
\end{aligned}$$

The above equation has a nonnegative real solution $\beta = 1/(1-2p)$ if and only if $0 < p < 1/2$. Putting this together with the result from part b) this implies that $(X_n)_{n\in\mathbb{N}_0}$ is null recurrent if $p = 1/2$. Finally, we know from an example from the lecture that there exists a stationary distribution for $(X_n)_{n\in\mathbb{N}_0}$ if $0 < p < 1/2$. Thus we may deduce from that $(X_n)_{n\in\mathbb{N}_0}$ is positive recurrent if $0 < p < 1/2$.

(d) For $p < 1/2$, there exists a reversible distribution and therefore stationary for the reflected random walk. By a theorem of the lecture this stationary distribution π is unique and given by

$$\begin{aligned}
\pi(0) &= \frac{q-p}{2q}, \\
\pi(n) &= \left(\frac{p}{q}\right)^{n-1} \frac{q-p}{2q^2}, \text{ for } n \in \mathbb{N}\backslash\{0\}.
\end{aligned}$$

Solution 10.2

a) We define for $x \neq y$

$$r'_{x,y} = r_{x,y}\left(\frac{\pi(y)r_{y,x}}{\pi(x)r_{x,y}} \wedge 1\right)$$

and

$$r'_{x,x} = r_{x,x} + \sum_{y\in E\backslash\{x\}} r_{x,y}\left(1 - \left(\frac{\pi(y)r_{y,x}}{\pi(x)r_{x,y}} \wedge 1\right)\right).$$

Clearly $r'_{x,y} \geq 0$ for all $x, y \in E$. Furthermore

$$\begin{aligned}
\sum_{y\in E} r'_{x,y} &= \sum_{y\in E\backslash\{x\}} r_{x,y}\left(\frac{\pi(y)r_{y,x}}{\pi(x)r_{x,y}} \wedge 1\right) + r_{x,x} + \sum_{y\in E\backslash\{x\}} r_{x,y}\left(1 - \left(\frac{\pi(y)r_{y,x}}{\pi(x)r_{x,y}} \wedge 1\right)\right) \\
&= \sum_{y\in E} r_{x,y} = 1,
\end{aligned}$$

hence $(r'_{x,y})_{x,y\in E}$ is a transition probability.

b) We have to show that for all $x, y \in E$ we have

$$\pi(x) r'_{x,y} = \pi(y) r'_{y,x}.$$

Let $x \neq y \in E$. If $\frac{\pi(y) r_{y,x}}{\pi(x) r_{x,y}} < 1$, then $\frac{\pi(x) r_{x,y}}{\pi(y) r_{y,x}} \geq 1$, hence

$$\pi(x) r'_{x,y} = \pi(x) r_{x,y} \frac{\pi(y) r_{y,x}}{\pi(x) r_{x,y}} = \pi(y) r_{y,x} \underbrace{\left(\frac{\pi(x) r_{x,y}}{\pi(y) r_{y,x}} \wedge 1 \right)}_{=1} = \pi(y) r'_{y,x}.$$

If $\frac{\pi(y) r_{y,x}}{\pi(x) r_{x,y}} \geq 1$, then $\frac{\pi(x) r_{x,y}}{\pi(y) r_{y,x}} < 1$, hence

$$\pi(x) r'_{x,y} = \pi(x) r_{x,y} = \underbrace{\frac{\pi(x) r_{x,y}}{\pi(y) r_{y,x}}}_{=\left(\frac{\pi(x) r_{x,y}}{\pi(y) r_{y,x}} \wedge 1 \right)} \pi(y) r_{y,x} = \pi(y) r'_{y,x}.$$

Remark. The Metropolis-Hastings algorithm is useful for generating samples from a distribution π in case only the relative likelihoods $\pi(y)/\pi(x)$ are known (but the normalizing constant is difficult to compute). After a burn-in period, the simulated Markov chain is assumed to be close to the stationary (reversible) distribution π, and samples can be drawn. Care has to be taken due to the serial correlation in this Markov chain, e.g. by only using every n-th sample, where the jump size n is large.

Solution 10.3

(a) Note that we have

$$\nu_i r_{i,j} = \nu_j r_{j,i}.$$

Summing up over i gives

$$\sum_{i \in \mathbb{N}_0} \nu_i r_{i,j} = \sum_{i \in \mathbb{N}_0} \nu_j r_{j,i} = \nu_j \sum_{i \in \mathbb{N}_0} r_{j,i} = \nu_j.$$

(b) Note that the birth and death process is irreducible as all states are connected. Due to the statement proved in Exercise 9.2 (b) it follows from $\nu_i r_{i,j} = \nu_j r_{j,i}$ that the chain is reversible.