ETH Zürich FS 2015
D-MATH
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Brownian Motion and Stochastic Calculus
Exercise Sheet 1

Exercise 1-1

Let $(\Omega, \mathcal{F}, P)$ be a probability space and assume that $X = (X_t)_{t\geq 0}$, $Y = (Y_t)_{t\geq 0}$ are two stochastic processes on $(\Omega, \mathcal{F}, P)$. Recall that two processes Z and Z' on $(\Omega, \mathcal{F}, P)$ are said to be *versions* (or *modifications*) of each other if $P(Z_t = Z'_t) = 1 \; \forall \; t \geq 0$, while Z and Z' are *indistinguishable* if $P(Z_t = Z'_t \; \forall \; t \geq 0) = 1$.

a) Assume that X and Y are both right-continuous or left-continuous. Show that the processes are versions of each other if and only if they are indistinguishable.

Remark: A stochastic process is said to *have the path property* $\mathcal{P}$ ($\mathcal{P}$ can be continuity, right-continuity, differentiability, boundedness...) if the property $\mathcal{P}$ holds for P-almost every path.

b) Give an example showing that one of the implications of part **a)** does not hold for general X, Y.

Exercise 1-2

Let $X = (X_t)_{t\geq 0}$ be a stochastic process defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$. The aim of this exercise is to show the following chain of implications:

$$X \text{ optional} \Rightarrow X \text{ progressively measurable} \Rightarrow X \text{ product-measurable and adapted.}$$

1. Show that every progressively measurable process is product-measurable and adapted.
2. Assume that X is adapted and *every* path of X is right-continuous. Show that X is progressively measurable.
   Remark: The same conclusion holds true if every path of X is left-continuous.
   Hint: For fixed $t \geq 0$, consider an approximating sequence of processes Y^n on $\Omega \times [0, t]$ given by $Y^n_0 = X_0$ and $Y^n_u = \sum_{k=0}^{2^n-1} 1_{(tk2^{-n}, t(k+1)2^{-n}]}(u) X_{t(k+1)2^{-n}}$ for $u \in (0, t]$.
3. Recall that the optional σ-field $\mathcal{O}$ is generated by the class $\overline{\mathcal{M}}$ of all adapted processes whose paths are all RCLL. Show that $\mathcal{O}$ is also generated by the subclass $\mathcal{M}$ of all *bounded* processes in $\overline{\mathcal{M}}$.
4. Use the monotone class theorem to show that every optional process is progressively measurable.

Exercise 1-3

Consider two random variables X and Y on a probability space $(\Omega, \mathcal{F}, P)$. Show that $X = Y$ a.s. if and only if

$$E[f(X)g(Y)] = E[f(X)g(X)] \tag{1}$$

for all bounded continuous real valued functions f and g.
Hint: You can use the monotone class theorem to show first that $E[h(X, Y)] = E[h(X, X)]$ for all bounded measurable functions h on $\mathbb{R} \times \mathbb{R}$.

Exercise 1-4

MATLAB-Exercise The aim of this exercise is to illustrate Donsker's Theorem, i.e., we consider a *rescaled* random walk S_n and let n goes to infinity. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(Y_k)_{k\in\mathbb{N}}$ be a sequence of i.i.d random variables where each Y_k is either 1 or -1, with a 50% probability for either value. Moreover, set $S_0 = 0$ and define $S_n = \sum_{k=1}^n Y_k$ for $n \in \mathbb{N}$. For $t \in [0,1]$ and $\omega \in \Omega$, consider the piecewise linear interpolation $X^n = (X_t^n)_{0\le t\le 1}$ with

$$X_t^n(\omega) = \frac{1}{\sqrt{n}} S_{[nt]}(\omega) + \frac{1}{\sqrt{n}} Y_{[nt]+1}(\omega)(nt - [nt]),$$

where $[x]$ denotes the integer part of the real number x.

For $n = 10^6$ simulate the random variables $(Y_k)_k$ and plot the resulting process $X_\cdot^n(\omega)$ (you can set the time grid $dt = 10^{-4}$).

Hint: The MATLAB commands, *binornd, cumsum, fix* might be useful.

Exercise sheets and further information are also available on:

http://www.math.ethz.ch/education/bachelor/lectures/fs2015/math/bmsc