ETH Zürich FS 2015
D-MATH
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Brownian Motion and Stochastic Calculus
Exercise Sheet 10

1. Consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ satisfying the usual conditions and let W be a Brownian motion with respect to P and $(\mathcal{F}_t)_{t\geq 0}$. Moreover, let $b \in L^2_{\text{loc}}(W)$ and assume there is a sequence $(t_n)_{n\in\mathbb{N}_0} \subset \mathbb{R}$ increasing to infinity, with $t_0 = 0$, such that

$$E\left[\exp\left(\frac{1}{2}\int_{t_n}^{t_{n+1}} b_s^2\, ds\right)\right] < \infty$$

for each $n \in \mathbb{N}_0$.

a) Show that $Z := \mathcal{E}(\int b\, dW)$ is a martingale.
Hint: Use Ex 8-2b). For that, note that it suffices to show that $n \mapsto E[Z_{t_n}]$ is constant, and prove this using induction and Novikov's criterion.

b) Show that the process $M = (M_t)_{t\geq 0}$ given by

$$M_t = \left(W_t - \int_0^t b_s\, ds\right) Z_t$$

is a martingale.
Hint: Use part a), the Girsanov transformation and Bayes' formula.

2. Consider a probability space $(\Omega, \mathcal{F}, P)$ carrying a Brownian motion $W = (W_t)_{t\geq 0}$. Denote by $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ the P-augmentation of the (raw) filtration generated by W. Moreover, fix $T > 0$, $a < b$, and set $F := 1_{\{a\leq W_T\leq b\}}$. The goal of this exercise is to find explicitly the integrand $H \in L^2_{\text{loc}}(W)$ in the Itô representation

$$F = E[F] + \int_0^\infty H_s\, dW_s. \qquad (\star)$$

a) Show that the martingale $M = (M_t)_{t\geq 0}$ given by $M_t := E[F|\mathcal{F}_t]$ has the representation

$$M_t = g(W_t, t), \quad 0 \leq t \leq T,$$

Bitte wenden!

for a Borel function $g : \mathbb{R} \times [0, T) \to \mathbb{R}$. Compute g in terms of the distribution function Φ of the standard normal distribution.
Hint: Use the Markov property of BM or the independent increment property of BM.

b) Apply Itô's formula to $g(W_t, t)$.
Hint: Since M is a martingale, you do not need to calculate all the terms in Itô's formula.

c) From part b), deduce a candidate for H and show that it works for Itô's representation of F in $(\star)$.

3. Let $T > 0$ denote a fixed time horizon and let $N = (N_t)_{t\in[0,T]}$ be a Poisson process with parameter $\lambda > 0$ on the probability space $(\Omega, \mathcal{F}, P)$ with respect to a filtration $\mathcal{F} = (\mathcal{F}_t)_{t\in[0,T]}$. Fix $\widetilde{\lambda} > 0$ and set $D := e^{(\lambda-\widetilde{\lambda})T} \left(\frac{\widetilde{\lambda}}{\lambda}\right)^{N_T}$.

a) Show that there exists a probability measure Q which is equivalent to P on $\mathcal{F}_T$ such that $\frac{dQ}{dP} = D$. Moreover, find its density process $Z = (Z_t)_{t\in[0,T]}$, and show that Z satisfies $dZ_t = \frac{\widetilde{\lambda}-\lambda}{\lambda} Z_{t-} d\widetilde{N}_t$, where $\widetilde{N}_t := N_t - \lambda t$, $t \in [0, T]$, denotes the compensated Poisson process.
Hint: Use Ex 9-3.

b) Show that P-a.s. for all $t \in [0, T]$, we have

$$\int_0^t \frac{1}{Z_s} d[Z, \widetilde{N}]_s = \frac{\widetilde{\lambda} - \lambda}{\widetilde{\lambda}} N_t.$$

Hint: Compute $\frac{Z_{s-}}{Z_s} \Delta N_s$ for $s \in (0, t]$.

c) Use Girsanov's Theorem to show that $N_t - \widetilde{\lambda} t$, $t \in [0, T]$, is a local $(Q, \mathcal{F})$-martingale.

4. Let $\mathcal{H}^1$ denote the space of all RCLL martingales with finite $\mathcal{H}^1$ norm. Recall that $\|M\|_{\mathcal{H}^1} := \mathbb{E}[\sup_{t\geq 0} |M_t|]$. The goal of this exercise is to show that the $\mathcal{H}^1$ space is complete.

a) Let $\Omega' := \{f : [0, \infty) \times \Omega \mapsto \mathbb{R}, \text{such that f has RCLL sample path}\}$ and on this space we define $\|f\|_{\mathcal{L}^1} := \mathbb{E}[\|f\|_\infty] := \mathbb{E}[\sup_{t\geq 0} |f(t)|]$. Show that

$$\mathcal{L}^1(\Omega; \mathcal{D}[0, \infty)) := \{f \in \Omega' : \|f\|_{\mathcal{L}^1} < \infty\}$$

is a Banach space.

Siehe nächstes Blatt!

b) Use part (a) to conclude that $\mathcal{H}^1$ is complete.

5. Matlab Exercise Let W be a one-dimensional $(P, \mathcal{F})$ Brownian motion on $[0, 1]$ and let $a(s) = s$ and $f(s) = cos(s)$. The goal of this exercise is to compute $\mathbb{E}_P[f(W_1 + \int_0^1 a_s ds)]$ by using Girsanov's Theorem. That is, we want to numerically verify the identity

$$\mathbb{E}_P\left[f(W_1 + \int_0^1 a_s ds)\right] = \mathbb{E}_P\left[\exp\left(\int_0^1 a_s dB_s - \frac{1}{2}\int_0^1 a_s^2 ds\right) f(W_1)\right]. \tag{1}$$

Simulate both sides of (1) using Monte Carlo simulation and verify that it is indeed true. For your Monte-Carlo simulation take $N = 10^4$ sample paths and 10^3 grid points on $[0, 1]$.
Hint: To approximate the stochastic integral $\int_0^1 a_s dB_s$ you can use Ex 8-4.