ETH Zürich FS 2015
D-MATH
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Brownian Motion and Stochastic Calculus
Exercise Sheet 3

1. Let $(B_t)_{t\geq 0}$ be a Brownian motion and consider the process X defined by

$$X_t := e^{-t} B_{e^{2t}}\,, \quad t \in \mathbb{R}.$$

a) Show that $X_t \sim \mathcal{N}(0,1), \quad \forall t \in \mathbb{R}$.

b) Show that the process $(X_t)_{t\in\mathbb{R}}$ is *time reversible*, i.e. $(X_t)_{t\geq 0} \stackrel{Law}{=} (X_{-t})_{t\geq 0}$.

Hint: Use the time inversion property of Brownian motion, i.e., if W is a Brownian motion, then

$$X_t := \begin{cases} 0, & \text{if } t = 0, \\ tW_{1/t}, & \text{if } t > 0, \end{cases}$$

is also a Brownian motion.

2. Let $X = (X_t)_{t\geq 0}$ be a right-continuous martingale on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ and let $\mathcal{F}_\infty := \sigma\Big(\bigcup_{t\geq 0} \mathcal{F}_t\Big)$. Show that the following are equivalent:

1) There is a random variable $Y \in \mathcal{L}^1$ with $X_t = E[Y|\mathcal{F}_t]$ for all $t \geq 0$.

2) $(X_t)_{t\geq 0}$ converges in $\mathcal{L}^1$ to an $\mathcal{F}_\infty$-measurable random variable.

3) There is an $\mathcal{F}_\infty$-measurable random variable $X_\infty \in \mathcal{L}^1$ such that $(X_t)_{t\in[0,\infty]}$ is a martingale.

4) $(X_t)_{t\geq 0}$ is uniformly integrable.

Moreover, show that if **1) – 4)** hold true, then $X_\infty = E[Y|\mathcal{F}_\infty]$.

Hint:

- For this exercise the *supermartingale convergence theorem* might be useful: If $X = (X_t)_{t\geq 0}$ is a right-continuous supermartingale with $\sup_{t\geq 0} E[|X_t|] < \infty$, then the limit $X_\infty := \lim_{t\to\infty} X_t$ exists a.s. (and $X_\infty \in \mathcal{L}^1$)

- For the last claim, first show that $\mathcal{D} := \{A \in \mathcal{F} | E[X_\infty \mathbf{1}_A] = E[Y\mathbf{1}_A]\}$ is a Dynkin-System which contains $\cup_{t\geq 0}\mathcal{F}_t$. In a second step use Dynkin System Theorem to conclude the result.

3. Let $S = C[0,1]$ be the space of continuous functions $x : [0,1] \to \mathbb{R}$ with the sup-norm $\|x\| := \sup_{0\leq t\leq 1} |x(t)|$ and the corresponding metric $d(x,y) := \|x-y\|$. Then, S is a Banach space and separable, since $[0,1]$ is compact. We consider the following sigma-algebras on S:

- the Borel sigma-algebra $\mathcal{B}(S)$
- the sigma-algebra $\sigma(C(S))$ generated by continuous functions $f : S \mapsto \mathbb{R}$, i.e.,

$$\sigma(C(S)) := \sigma\left(\bigcup_{f\in C(S)} \{f^{-1}(B), B \in \mathcal{B}(\mathbb{R})\}\right)$$

- the sigma algebra $\sigma(\mathcal{Z})$ generated by the system $\mathcal{Z}$ of all *cylinder sets*

$$Z = \{x \in S | x(t_j) \in A_j, j = 1, \dots, n\},$$

  with $n \in \mathbb{N}, 0 \leq t_1 < t_2 < \cdots < t_n \leq 1$ and $A_1, \dots, A_n \in \mathcal{B}(\mathbb{R})$.

Show that

$$\sigma(C(S)) = \mathcal{B}(S) = \sigma(\mathcal{Z}).$$

That is, we have the following observation. The map $X : (\Omega, \mathcal{F}) \mapsto (S, \mathcal{B}(S))$ with $\omega \mapsto X_\cdot(\omega)$ is measurable if and only if for all $t_i \in [0,1]$ the coordinate evaluation map $X_{t_i} : (\Omega, \mathcal{F}) \mapsto (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ with $\omega \mapsto X_{t_i}(\omega)$ is measurable.
Hint:

- To show $\sigma(C(S)) \supseteq \mathcal{B}(S)$: let $A \subseteq S$ be closed and find a function $h \in C(S)$ such that $A = \{h = 0\}$.
- To show $\mathcal{B}(S) \subseteq \sigma(\mathcal{Z})$: first show that every closed ε-ball is in $\sigma(\mathcal{Z})$ and use separability of the metric space S to conclude.

4. **Matlab Exercise** The goal of this exercise is illustrate the Wiener-Lévy representation of Brownian motion. Therefore, for $n \in \mathbb{N}$ let $\varphi_{n,k}$ and φ_0 denote the Schauder functions, i.e.,

$$\begin{aligned} \varphi_0(t) &:= t \\ \varphi_{n,k}(t) &:= 2^{n/2}(t-(k-1)2^{-n})I_{J_{2k-1,n+1}} - 2^{n/2}(t-k2^{-n})I_{J_{2k,n+1}}(t), \end{aligned}$$

Siehe nächstes Blatt!

where $I_A(t)$ denotes the indicator function on A and

$$J_{k,n} = ((k-1)2^{-n}, k2^{-n}], \quad \text{for} \quad k = 1, \ldots, 2^n.$$

That is, the graph of $\varphi_{n,k}$ is a triangle over $J_{k,n}$ with its peak of height $2^{-n/2-1}$ at the middle point $(2k-1)2^{-(n+1)}$. Moreover, let Y_0 and $Y_{n,k}$ be i.i.d standard normal random variables and define for $N \leq \infty$

$$W_t^N := Y_0\varphi_0(t) + \sum_{n=0}^{N}\sum_{k=1}^{2^n} Y_{n,k}\varphi_{n,k}(t).$$

We know from the lecture that W^∞ is well-defined and is a Brownian motion.
Simulate 10 sample paths of the process W^N with $N = 12$. In this exercise you can set $T = 1$ and use an equidistant time grid with 2000 grid points, i.e., $t_i = i/M, i = 0, \ldots, M = 2 \cdot 10^3$.
Hint:

- First write a function *schauderfunction(n,k,t)* which computes the schauder functions for given n, k and t
- Figure out how many iid normal random variables you need and compute W^N by sequentially adding the new increments