ETH Zürich FS 2015
D-MATH
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Brownian Motion and Stochastic Calculus
Exercise Sheet 9

1. Let $M := (M_t)_{t\geq 0}$ be a continuous martingale of finite variation. Show that

$$P\text{-a.s.},\ \forall t \geq 0,\ M_t = M_0.$$

Hint: First, consider the case where M has uniformly bounded variation and show that $\mathbb{E}[M_t^2] = 0$. Then, use a suitable stopping time T and consider the stopped process $M^T := (M_{t\wedge T})_{t\geq 0}$.

2. Let $W = (W_t)_{t\geq 0}$ be a 1-dimensional Brownian motion.

a) Prove that for every polynomial p on $\mathbb{R}$, the stochastic integral $\int p(W)dW$ is well defined. Moreover, show that $\int p(W)dW$ is also a martingale.

Hint: Use Ex 7-2 a).

b) Show that the process $X = (X_t)_{t\geq 0}$ given by $X_t = e^{\frac{1}{2}t}\cos W_t$ is a martingale.

Hint: Apply Itô's formula.

c) Let W' be another Brownian motion independent of W and ϱ be an adapted, left-continuous process satisfying $|\varrho| \leq 1$. Prove that the process $B = (B_t)_{t\geq 0}$ given by

$$B_t = \int_0^t \varrho_s\, dW_s + \int_0^t \sqrt{1-\varrho_s^2}\, dW'_s$$

is a Brownian motion. Moreover, compute $[B, W]$.
Hint: Use Lévy's characterization of Brownian motion.

3. Let $N = (N_t)_{t\geq 0}$ be a Poisson process with parameter $\lambda > 0$ with respect to a probability measure P and a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$. Recall that a *Poisson process with parameter* $\lambda > 0$ *w.r.t.* P *and* $\mathcal{F}$ is a (real-valued) stochastic process $N = (N_t)_{t\geq 0}$ which is adapted to $\mathcal{F}$, starts at 0 (i.e. $N_0 = 0$ P- a.s. .) and satisfies the following two properties:

Bitte wenden!

(PP1) For $0 \leq s < t$, the *increment* $N_t - N_s$ is independent (under P) of $\mathcal{F}_s$ and is (under P) *Poisson-distributed with parameter* $\lambda(t-s)$, i.e.

$$P[N_t = k] = \frac{(\lambda(t-s))^k}{k!} e^{-\lambda(t-s)}, \quad k \in \mathbb{N}_0.$$

(PP2) N is a *counting process* with jumps of size 1, i.e. for P-almost all ω, the function $t \mapsto N_t(\omega)$ is right-continuous with left limits (RCLL), piecewise constant and $\mathbb{N}_0$-valued, and increases by jumps of size 1.

Let $\widetilde{\lambda} > 0$ and define $S_t := e^{(\lambda - \widetilde{\lambda})t} \left(\frac{\widetilde{\lambda}}{\lambda}\right)^{N_t}$.

a) Show that we have $P-$ a.s. for all $t > 0$

$$\Delta S_t = \frac{\widetilde{\lambda} - \lambda}{\lambda} S_{t-} \Delta N_t \,.$$

b) Show that P- a.s. for all $t \geq 0$, we have

$$S_t = 1 + \int_0^t \frac{\widetilde{\lambda} - \lambda}{\lambda} S_{u-} \, \mathrm{d}\widetilde{N}_u \,,$$

where $\widetilde{N}_t := N_t - \lambda t$, $t \geq 0$, denotes the compensated Poisson process.

Hint: Write $S_t = f(t, N_t)$ and apply Itô's formula.

c) Deduce that S is a local $(P, \mathcal{F})$-martingale. Show that it is even a true $(P, \mathcal{F})$-martingale.

Hint: Show that $\sup_{0 \leq t \leq T} |S_t|$ is integrable for each $T > 0$.

4. Consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ satisfying the usual conditions and let $\sigma \leq \tau$ be two stopping times. Moreover, let Z be a bounded, $\mathcal{F}_\sigma$-measurable random variable. The goal of this exercise is to compute the stochastic integral process $\int Z 1_{]\!]\sigma,\tau]\!]} \, dM$ for an integrator $M \in \mathcal{M}^c_{0,\text{loc}}$.

a) For a (uniformly integrable) right-continuous martingale $X = (X_t)_{t \geq 0}$, show that the process $Z(X^\tau - X^\sigma)$ is again a (uniformly integrable) right-continuous martingale.

Hint:

(i) The following result might be helpful: suppose $N = (N_t)_{0 \leq t \leq \infty}$ is an adapted, right-continuous process with the property that for any stopping time τ, we have $N_\tau \in L^1(P)$ and $\mathbb{E}[N_\tau] = \mathbb{E}[N_0]$. Then N is a uniformly integrable martingale. If N is only defined on $[0, \infty)$ and we have the above assumption on N_τ only for bounded or finite stopping times τ, then N is still a martingale but might not be UI.

Siehe nächstes Blatt!

(ii) Use (i) to show the assertion for $Z = 1_A$ for some $A \in \mathcal{F}_\sigma$ and extend the result to general Z using measure theoretical induction.

b) Let $M, N \in \mathcal{M}^c_{0,\text{loc}}$. Show that

$$[Z(M^\tau - M^\sigma), N] = Z[M^\tau - M^\sigma, N] = Z([M, N]^\tau - [M, N]^\sigma).$$

Hint: Use the fact that we have for all stopping times τ

$$[M^\tau, N] = [M, N^\tau] = [M, N]^\tau$$

and part a).

c) Let $M \in \mathcal{M}^c_{0,\text{loc}}$ and set $H := Z 1_{]\!]\sigma,\tau]\!]}$. Show that the stochastic integral $\int H dM$ is well-defined and

$$\int H\, dM = Z(M^\tau - M^\sigma).$$

Conclude with part a) that if M is a (uniformly integrable) martingale, then the stochastic integral $\int H\, dM$ is also a (uniformly integrable) martingale.

Hint:

- To show H is adapted use the fact that for two stopping times σ and ϱ we have that if $A \in \mathcal{F}_\sigma$ then $A \cap \{\sigma < \varrho\}$ belong to $\mathcal{F}_\varrho$
- Use part b) to compute $[Z(M^\tau - M^\sigma), N]$ for $N \in \mathcal{M}^c_{0,\text{loc}}$