Brownian Motion and Stochastic Calculus
Sketch of Solution Sheet 1

1. **a)** We just show that the fact that X is a version of Y implies the indistinguishability, since the converse is obvious. Without loss of generality, we assume that X and Y are right-continuous.

For $t \geq 0$, we define the null set $N_t := \{\omega : X_t(\omega) \neq Y_t(\omega)\}$. We consider $N := \cup_{t\in\mathbb{Q}_+} N_t$, which remains a null set as a countable union of null sets. Finally, we introduce the null set $A_Z := \{\omega : Z_\cdot(\omega) \text{ not right-continuous}\}$ for $Z = X, Y$ and we define $M := A_X \cup A_Y \cup N$, which is still a null set.

It suffices to check that, for all $\omega \in M^c$, $X_t(\omega) = Y_t(\omega)\ \forall\, t \geq 0$. By definition of M we clearly have that, for $\omega \in M^c$, $X_t(\omega) = Y_t(\omega)\ \forall\, t \in \mathbb{Q}_+$. Now, take any $t \geq 0$ and let (t_n) be a sequence in $\mathbb{Q}_+$ with $t_n \downarrow t$. The right-continuity of the paths $X_\cdot(\omega)$ and $Y_\cdot(\omega)$ then implies $X_t(\omega) = \lim_{n\to\infty} X_{t_n}(\omega) = \lim_{n\to\infty} Y_{t_n}(\omega) = Y_t(\omega)$.

b) Take $\Omega = [0,\infty)$, $\mathcal{F} = \mathcal{B}([0,\infty))$ the Borel σ-algebra, and P a probability measure with $P(\{\omega\}) = 0\ \forall\, \omega \in \Omega$ (for instance, the exponential distribution).

Set $X \equiv 0$ and $Y_t(\omega) = \begin{cases} 1, & t = \omega, \\ 0, & \text{else.} \end{cases}$

Then, $P[X_t = Y_t] = 1\ \forall\, t \geq 0$, since single points have no mass, but $\{X_t = Y_t\ \forall\, t \geq 0\} = \emptyset$. Note that all sample paths of X are continuous, while all sample paths of Y are discontinuous at $t = \omega$.

2. **a)** Let X be progressively measurable. Then $X|_{\Omega\times[0,t]}$ is $\mathcal{F}_t \otimes \mathcal{B}[0,t]$-measurable for every $t \geq 0$. For any $t \geq 0$, we see that $X_t = X \circ i_t$, where $i_t : (\Omega, \mathcal{F}_t) \to (\Omega \times [0,t], \mathcal{F}_t \otimes \mathcal{B}[0,t])$, $\omega \mapsto (\omega, t)$ is measurable. Therefore, X_t is $\mathcal{F}_t$-measurable for every $t \geq 0$. Moreover, the processes X^n defined by $X^n_u := X|_{\Omega\times[0,n]} 1_{[0,n]}(u)$, $u \geq 0$, are $\mathcal{F} \otimes \mathcal{B}[0,\infty)$-measurable. Since $X^n \to X$ pointwise (in (t,ω)) as $n \to \infty$, also X is $\mathcal{F} \otimes \mathcal{B}[0,\infty)$-measurable.

b) Fix a $t \geq 0$ and consider the sequence of processes Y^n on $\Omega \times [0,t]$ given by $Y^n_0 = X_0$ and $Y^n_u = \sum_{k=1}^{2^n-1} 1_{(tk2^{-n}, t(k+1)2^{-n}]}(u) X_{t(k+1)2^{-n}}$ for $u \in (0,t]$. By construction, each Y^n is $\mathcal{F}_t \otimes \mathcal{B}[0,t]$-measurable. Since $Y^n \to X|_{\Omega\times[0,t]}$ pointwise as $n \to \infty$ due to right-continuity, the result follows.

c) Let X be adapted, with all paths being RCLL. Consider the processes $X^n := (X \wedge n) \vee (-n)$. Clearly, each X^n is bounded and RCLL. Thus, each X^n is $\sigma(\mathcal{M})$-measurable. As the pointwise limit of the X^n, also X is $\sigma(\mathcal{M})$-measurable. It follows that $\mathcal{O} \subset \sigma(\mathcal{M})$. The reverse inclusion is trivial.

d) If a process X is optional, then $X^n := X\, 1_{\{|X| \le n\}}$ is also optional and of course $X^n \to X$; so if each X^n is progressively measurable, then so is X, and hence we can assume without loss of generality that X is bounded.
Let $\mathcal{H}$ denote the real vector space of bounded, progressively measurable processes. By part b), $\mathcal{H}$ contains $\mathcal{M}$. Clearly, $\mathcal{H}$ contains the constant process 1 and is closed under monotone bounded convergence. Also, $\mathcal{M}$ is closed under multiplication. The monotone class theorem yields that every bounded $\sigma(\mathcal{M})$-measurable process is contained in $\mathcal{H}$. Due to c), we conclude that every bounded optional process is progressively measurable.

3. We only show the 'if' part of the claim, since the other part is trivial. Let $\mathcal{H}$ be the set of all bounded measurable functions $h(x,y)$ on $\mathbb{R} \times \mathbb{R}$ such that $E[h(X,X)] = E[h(X,Y)]$. Then, $\mathcal{H}$ contains the constant 1 function and is a vector space closed under bounded monotone convergence. Let $\mathcal{D}$ be the set (closed under multiplication) of all functions of the form $(x,y) \mapsto f(x)g(y)$, where f and g are bounded continuous function of $\mathbb{R}$. Moreover, Formula (1) shows that $\mathcal{D} \subset \mathcal{H}$ and we know that $\sigma(\mathcal{D}) = \mathcal{B}(\mathbb{R}^2)$. The monotone class theorem implies that all bounded measurable functions on $\mathbb{R}^2$ are in $\mathcal{H}$. Now take the indicator of the diagonal: $h(x,y) = 1$ if $x = y$ and 0 otherwise, yielding that $X = Y$ a.s.

4. Matlab Files

```
function bmscex14
clear all;
% EX 1-4
% Illustration of Donsker's Theorem.

% number of increments
n=10^6;

% simulate n bernoulli variables
p=1/2;
ber= binornd(1,p,n,1);

```

Siehe nächstes Blatt!

```
% linear transformation which makes [1,0]--> [1,-1]
y= 2*ber -1;

% sum S with S_0=0
sumprocess= [0;cumsum(y)];

%timestep for the plot
dt=10^(-4);
%time grid
timegrid = 0:dt:1;
t=timegrid(1:(end-1));

% use the linear interpolation formula given in the exercise
% indices are shifted by 1, since we start with at time 0
x=1/sqrt(n)*sumprocess(fix(n*t)+1)+1/sqrt(n)*y(fix(n*t)+1).*(n*t-fix(n*t))';

%add terminal value x_1=1/sqrt(n)*S_n
x= [x;1/sqrt(n)*sumprocess(fix(n)+1)];

%plot options
plot(timegrid,x,'r-')
title('Ex 1-4: Donskers Theorem')
xlabel('time');
ylabel('rescaled random walk');
legend(strcat('number of increments=', num2str(n)));
end
```

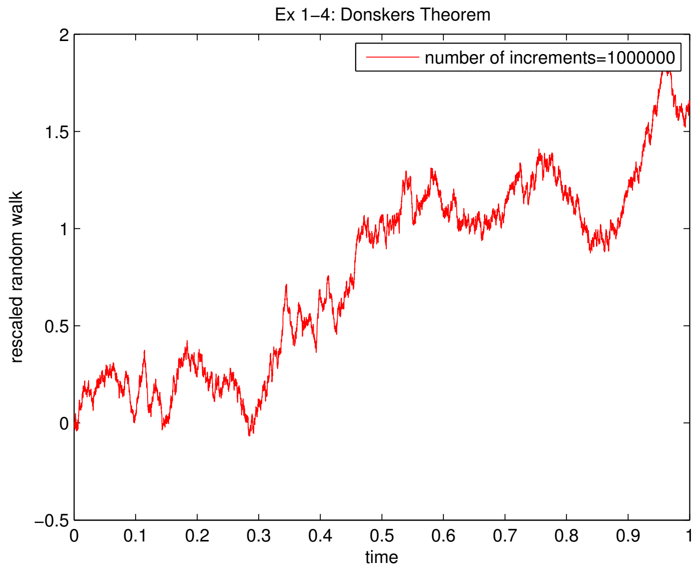

Abbildung 1: Illustration of Donsker's Theorem