ETH Zürich FS 2015
D-MATH
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Brownian Motion and Stochastic Calculus
Sketch of Solution Sheet 10

1. **a)** Being a nonnegative local martingale with $Z_0 = 1$, $Z := \mathcal{E}(\int b\,dW)$ is a supermartingale and hence has a nonincreasing mean function. To show that it is a martingale, it thus suffices to show that $E[Z_{t_n}] = E[Z_{t_0}] = 1$ for all n. Fix $n \geq 1$ and set $b^n := b1_{]\!]t_{n-1},t_n]\!]}$. Then

$$E\left[\exp\left(\frac{1}{2}\left[\int b^n\,dW\right]_\infty\right)\right] = E\left[\exp\left(\frac{1}{2}\int_{t_{n-1}}^{t_n} b_s^2\,ds\right)\right] < \infty.$$

Therefore, Novikov's criterion yields that $\mathcal{E}(\int b^n\,dW)$ is a (uniformly integrable) martingale. Finally, noting that $Z_{t_n} = Z_{t_{n-1}}\mathcal{E}(\int b^n\,dW)_{t_n}$ and $\mathcal{E}(\int b^n\,dW)_{t_{n-1}} = 1$, we obtain

$$E[Z_{t_n}] = E\left[Z_{t_{n-1}}E\left[\mathcal{E}\left(\int b^n\,dW\right)_{t_n}\middle|\mathcal{F}_{t_{n-1}}\right]\right] = E[Z_{t_{n-1}}].$$

b) Fix $T \geq 0$ and $(Z^T)_t := Z_{t\wedge T}$ the stopped process. With $L := \int b\,dW$, we have $Z^T = \mathcal{E}(L^T)$. Since Z^T is a uniformly integrable martingale, Girsanov's theorem yields that

$$W - \langle L^T, W\rangle = W - \left\langle \int b1_{]\!]0,T]\!]}\,dW, W\right\rangle = W - \int b1_{]\!]0,T]\!]}\,dt$$

is a martingale (even a Brownian motion) under the measure Q^T given by $dQ^T = Z^T dP$. By Bayes' formula (cf. Proposition 4.(4.4) part 2), it follows that

$$\left(W - \int b1_{]\!]0,T]\!]}\,dt\right) Z^T$$

is a P-martingale. Hence also $M^T = \left(W^T - \int b1_{]\!]0,T]\!]}\,dt\right) Z^T$ is a P-martingale. Since T was arbitrary, M is a P-martingale.

2. **a)** By the Markov property of Brownian motion, we have for any $0 \leq t < T$,

$$M_t = E[1_{\{a\leq W_T\leq b\}}|\mathcal{F}_t] = K_{T-t}(W_t, [a,b])$$

where K is the Gaussian transition kernel. Define $g : \mathbb{R} \times [0,T) \to \mathbb{R}$ by

$$g(x,t) = K_{T-t}(x, [a,b]).$$

Then, denoting the standard normal distribution function by Φ, we have

$$g(x,t) = \Phi\left(\frac{b-x}{\sqrt{T-t}}\right) - \Phi\left(\frac{a-x}{\sqrt{T-t}}\right).$$

In particular, g is $C^{2,1}$ on $\mathbb{R} \times (0,T)$.

Alternative computation. Noting that W_t is $\mathcal{F}_t$-measurable and $W_T - W_t \sim \mathcal{N}(0, T-t)$ is independent of $\mathcal{F}_t$, we can compute

$$\begin{aligned} M_t = E[F|\mathcal{F}_t] &= P[a \le W_T \le b|\mathcal{F}_t] = P[a - W_t \le W_T - W_t \le b - W_t|\mathcal{F}_t] \\ &= \Phi\left(\frac{b-W_t}{\sqrt{T-t}}\right) - \Phi\left(\frac{a-W_t}{\sqrt{T-t}}\right) = g(W_t, t). \end{aligned}$$

b) Since $M_t = g(W_t, t)$ is a martingale, the sum of all finite variation terms in Itô's formula applied to $g(W_t, t)$ vanishes and we obtain for $t \in (0,T)$ that

$$\begin{aligned} M_t - M_0 &= \int_0^t \frac{\partial g}{\partial x}(W_s, s)\, dW_s \\ &= \int_0^t \frac{1}{\sqrt{T-s}} \left(\phi\left(\frac{a-W_s}{\sqrt{T-s}}\right) - \phi\left(\frac{b-W_s}{\sqrt{T-s}}\right)\right) dW_s, \end{aligned} \tag{1}$$

where $\phi = \Phi'$ denotes the standard normal density.

c) Since $x\phi(x) \to 0$ as $x \to \pm\infty$, it is easy to see that the integrand in (1) converges P-a.s. to 0 as $s \uparrow T$. Hence,

$$H := \frac{1}{\sqrt{T-s}} \left(\phi\left(\frac{a-W_s}{\sqrt{T-s}}\right) - \phi\left(\frac{b-W_s}{\sqrt{T-s}}\right)\right) 1_{[\![0,T[\![}$$

is a continuous, adapted process. Thus, $H \in L^2_{\mathrm{loc}}(W)$ and (1) yields for $0 \le t < T$ that

$$M_t = M_0 + \int_0^t H_s\, dW_s. \tag{2}$$

Since both sides in (2) are local martingales on $[0,\infty)$ and hence continuous, we can let $t \uparrow T$ to get

$$M_T = M_0 + \int_0^T H_s\, dW_s.$$

To conclude, it suffices to note that $M_T = F$, $M_0 = E[F]$, and $\int_0^T H_s\, dW_s = \int_0^\infty H_s\, dW_s$ since H is zero on $[\![T,\infty]\!]$. Moreover, $\int H\, dW$ is a martingale as M is one.

Siehe nächstes Blatt!

3. **a)** We know from Ex 9-3 that the process $Y := (Y_t)_{t\in[0,T]}$ defined by

$$Y_t := e^{(\lambda-\widetilde{\lambda})t}\left(\frac{\widetilde{\lambda}}{\lambda}\right)^{N_t} \tag{3}$$

is a true $(P,\mathcal{F})$-martingale which satisfies $\mathbb{E}_P[Y_t] = Y_0 = 1$ for all $0 \le t \le T$. Moreover, Y is clearly strictly positive. Hence $D = Y_T$ is a strictly positive $\mathcal{F}_T$-measurable random variable, which satisfies $\mathbb{E}_P[D] = 1$. Therefore, we can define a probability measure Q on $\mathcal{F}_T$ by $\mathrm{d}Q = D\,\mathrm{d}P$, which is equivalent to P. Moreover, its density process Z is given by

$$Z_t := \mathbb{E}_P[D|\,\mathcal{F}_t] = \mathbb{E}_P[Y_T|\,\mathcal{F}_t] = Y_t, \quad P.a.s. \tag{4}$$

for all $t \in [0,T]$. Therefore, in addition, we have shown in EX 9-3 b)

$$\mathrm{d}Z_t = \frac{\widetilde{\lambda}-\lambda}{\lambda} Z_{t-}\,\mathrm{d}\widetilde{N}_t. \tag{5}$$

b) Using that for P-almost all ω, we have $\Delta N_s \in \{0,1\}$ for all $s \in (0,T]$, we have $P.a.s.$ for all $s \in (0,T]$

$$\frac{Z_{s-}}{Z_s}\Delta N_s = \left(\frac{\lambda}{\widetilde{\lambda}}\right)^{\Delta N_s}\Delta N_s = \frac{\lambda}{\widetilde{\lambda}}\Delta N_s. \tag{6}$$

Recalling that we have $P.a.s.$ for all $t \in [0,T]$

$$[\widetilde{N}]_t = \sum_{0<s\le t}\Delta N_s^2 = \sum_{0<s\le t}\Delta N_s = N_t, \tag{7}$$

using the properties of the quadratic variation and (5), we get $P.a.s.$ for all $t \in [0,T]$

$$\begin{aligned}\int_0^t \frac{1}{Z_s}\,\mathrm{d}[Z,\widetilde{N}]_s &= \int_0^t \frac{\frac{\widetilde{\lambda}-\lambda}{\lambda}Z_{s-}}{Z_s}\,\mathrm{d}[\widetilde{N}]_s = \sum_{0<s\le t}\frac{\widetilde{\lambda}-\lambda}{\lambda}\frac{Z_{s-}}{Z_s}\Delta N_s \\ &= \sum_{0<s\le t}\frac{\widetilde{\lambda}-\lambda}{\lambda}\frac{\lambda}{\widetilde{\lambda}}\Delta N_s = \frac{\widetilde{\lambda}-\lambda}{\widetilde{\lambda}}\sum_{0<s\le t}\Delta N_s = \frac{\widetilde{\lambda}-\lambda}{\widetilde{\lambda}}N_t. \end{aligned}\tag{8}$$

c) By Girsanov's theorem and part b) it follows that

$$\begin{aligned}\widetilde{N}_t - \int_0^t \frac{1}{Z_s}\,\mathrm{d}[Z,\widetilde{N}]_s &= \widetilde{N}_t - \frac{\widetilde{\lambda}-\lambda}{\widetilde{\lambda}}N_t = N_t - \lambda t - N_t + \frac{\lambda}{\widetilde{\lambda}}N_t \\ &= \frac{\lambda}{\widetilde{\lambda}}(N_t - \widetilde{\lambda}t), \quad t \in [0,T], \end{aligned}\tag{9}$$

is a local $(Q,\mathcal{F})$-martingale. Since $\frac{\lambda}{\widetilde{\lambda}} \ne 0$ is a constant and since local martingales form a vector space, it follows that $N_t - \widetilde{\lambda}t$, $t \in [0,T]$, is a local $(Q,\mathcal{F})$-martingale, too.

Bitte wenden!

4. **a)** Since the $\mathcal{L}^1$ norm is a composition of two norms it is again a norm. To show completeness, it suffices to check that for any sequence

$$(f_n)_n \subseteq \mathcal{L}^1(\Omega; \mathcal{D}([0,\infty))$$

such that $\sum_{k=1}^{\infty} \|f_k\|_{\mathcal{L}^1} < \infty$ the limit $\lim_{n\to\infty} \sum_{k=1}^{n} f_k$ exists. With the triangle inequality we have

$$\|\sum_{j=1}^{n} \|f_j\|_\infty\|_{L^1(P)} \leq \sum_{j=1}^{n} \|\|f_j\|_\infty\|_{L^1(P)} \leq \sum_{j=1}^{\infty} \|f_j\|_{\mathcal{L}^1}.$$

Therefore, by monotone convergence theorem we have that

$$\|\sum_{j=1}^{\infty} \|f_j\|_\infty\|_{L^1(P)} < \infty.$$

In particular, we have $\sum_{j=1}^{\infty} \|f_j\|_\infty < \infty$ P-.a.s. Consequently, for $\omega \in \Omega$ and $t \geq 0$ the function

$$f(\omega, t) = \sum_{j=1}^{\infty} f_j(\omega, t)$$

exists a.s. and the series converges a.s. uniformly in t. In particular, f has RCLL trajectory. Finally, since

$$\|f(\omega) - \sum_{j=1}^{n} f_j(\omega)\|_\infty \leq \sum_{j\geq n+1} \|f_j(\omega)\|_\infty \leq \sum_{j=1}^{\infty} \|f_j(\omega)\|_\infty \in L^1(P).$$

The claim now follows from dominated convergence.

b) Now, let $(M^n)_{n\in\mathbb{N}} \subseteq \mathcal{H}^1$ be a Cauchy sequence. By construction, $(M^n)_{n\in\mathbb{N}} \subseteq \mathcal{L}^1(\Omega, \mathcal{D}([0,\infty)))$. Therefore, by part (a) there exists a $M \in \mathcal{L}^1(\Omega, \mathcal{D}([0,\infty)))$ such that $\|M^n - M\|_{\mathcal{L}^1} \to 0$ as $n \to \infty$. It remains to check that M is a martingale. Note that for all $t \geq 0$

$$\|M^n(t) - M(t)\|_{L^1(P)} \leq \|M^n(t) - M(t)\|_{\mathcal{L}^1} \to 0. \tag{10}$$

Hence, M is a martingale by Ex 3-2.

5. Matlab Files

```
function [expleft, expright]=bmsc104anew
% In this exercise we simulated the expectation of $cos(
    \int_0^1 s ds +
```

Siehe nächstes Blatt!

```
% B_1) where B is a P-BM by using Girsanov's Theorem, that is we check it
% is equal to E[exp(\int_0^1 s dB_s -1/2 \int_0^1 a_s^2 ds) cos (B_1)].
tic
%% parameter input
% horizon
T=1;
% sample size
Nplot=10^4;
% grid points
M=10^3;
% time step
dt= T/M;

%% Simulation
% BM
PBM = [zeros(1,Nplot);sqrt(T/M)*cumsum(randn(M,Nplot))];

% approximation of the integral \int_0^1 s dB_s (cf. EX 8-4)
agrid = repmat((0:dt:T)',1,Nplot);
Int = [zeros(1,Nplot);cumsum(agrid(1:(end-1),:).*(PBM(2:end,:)-PBM(1:(end-1),:)))];
% weights= exp(int_0^1 s d B_s-1/2 \int_0^1 s^2 ds
weights = (exp(Int(end,:)-1/6));

% LHS of (1)
expleft= mean(cos(PBM(end,:)+1/2));
% RHS of (1)
expright= mean(weights.* cos(PBM(end,:)));

toc
```