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Brownian Motion and Stochastic Calculus
Sketch of Solution Sheet 2

1. **a)** We need to show that for any $n \geq 1$ and any $0 \leq t_1 \leq t_2... \leq t_n \leq 1$ the random vector $(X_{t_1}, ..., X_{t_n})$ is a Gaussian vector. Fix any $n \geq 1$ and any $0 \leq t_1 \leq t_2... \leq t_n \leq 1$. It suffices to show that $(X_{t_1}, ..., X_{t_n})$ is the image of a linear transformation of another Gaussian vector. From Proposition 1.4 in the lecture notes, we know that Brownian motion W is a Gaussian process. We distinguish between two cases:

case 1: $t_n < 1$

In this case, the vector $(X_{t_1}, ..., X_{t_n})$ is the image of the Gaussian vector $(W_{t_1}, ..., W_{t_n}, W_1)$ under the linear map

A: $\mathbb{R}^{n+1} \to \mathbb{R}^n$ defined by $A := (a_{ij})$, $a_{ij} = \begin{cases} 1, & i = j \in \{1, ..., n\}, \\ -t_i, & j = n+1, \ i \in \{1, ..., n\} \\ 0 & \text{else.} \end{cases}$

case 2: $t_n = 1$

In that case, the vector $(X_{t_1}, ..., X_{t_n})$ is the image of the Gaussian vector $(W_{t_1}, ..., W_{t_{n-1}}, W_1)$ under the linear map B: $\mathbb{R}^n \to \mathbb{R}^n$ defined by $b_{ij} = \begin{cases} 1, & i = j \in \{1, ..., n-1\}, \\ -t_i, & j = n, \ i \in \{1, ..., n-1\} \\ 0 & \text{else.} \end{cases}$

In both cases, $(X_{t_1}, ..., X_{t_n})$ is the image of a linear transformation of a Gaussian vector, hence we are done.

For any $t \in [0, 1]$ we have

$$E[X_t] = E[W_t - t\, W_1] = 0.$$

For any $0 \leq s, t \leq 1$, using that $\operatorname{Cov}(W_t, W_s) = t \wedge s$ (see Proposition 1.1.4), we have

$$\begin{aligned} \operatorname{Cov}(X_t, X_s) &= \operatorname{Cov}(W_t - t\, W_1, W_s - s\, W_1) \\ &= \operatorname{Cov}(W_t, W_s) - s \operatorname{Cov}(W_t, W_1) - t \operatorname{Cov}(W_1, W_s) + ts \operatorname{Cov}(W_1, W_1) \\ &= t \wedge s - ts. \end{aligned}$$

b) Take any $t \in (0,1)$. We show that the increment $X_1 - X_t$, $X_t - X_0$ are correlated. In the same way as above we obtain that

$$\mathrm{Cov}(X_1 - X_t,\, X_t - X_0) = \mathrm{Cov}(-W_t + t\,W_1,\, W_t - t\,W_1) = t(t-1) \neq 0.$$

2. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables with $X_n \sim \mathcal{N}(\mu_n, \sigma_n^2)$ for each $n \in \mathbb{N}$. Since $(X_n)_{n\in\mathbb{N}}$ converges in probability to X, $(X_n - X)_{n\in\mathbb{N}}$ converges in probability to 0 and hence $(X_n - X)_{n\in\mathbb{N}}$ converges in distribution to 0.
Fix any $n \in \mathbb{N}$. The sequence $(X_n - X_k)_{k\in\mathbb{N}}$ converges in probability to $X_n - X$ and hence $(X_n - X_k)_{k\in\mathbb{N}}$ converges in distribution to $X_n - X$. Now, since by assumption $(X_n)_{n\in\mathbb{N}}$ is a *Gaussian process*, we get that for each k, $X_n - X_k$ is normal distributed. Thus, we deduce from the hint that $X_n - X$ is normal distributed. Since $n \in \mathbb{N}$ was arbitrarily chosen, we get that $(X_n - X)_{n\in\mathbb{N}}$ is a sequence of Gaussian random variables. Moreover, since $(X_n - X)_{n\in\mathbb{N}}$ converges in distribution to 0, we deduce again from the hint that

$$E[X_n - X] \longrightarrow 0 \quad \text{and} \quad \mathrm{Var}(X_n - X) \longrightarrow 0 \quad \text{as } n \to \infty.$$

As a consequence, we get directly the L^2 convergence of X^n to X, since

$$\|X_n - X\|_{L^2}^2 = E\big[|X_n - X|^2\big] = \big(E[X_n - X]\big)^2 + \mathrm{Var}\big(X_n - X\big).$$

3. Let $\tilde{P} = \bigotimes_{n=0}^{\infty} P_n$. By definition, $(X_n)_{n\geq 0}$ is independent with respect to P if and only if for all $n \in \mathbb{N}_0, A_0 \in \mathcal{S}_0, \dots, A_n \in \mathcal{S}_n$ we have

$$P[X_0 \in A_0, \dots, X_n \in A_n] = \Pi_{i=0}^{n} P[X_i \in A_i] = \Pi_{i=0}^{n} P_i[A_i].$$

On the other hand, the defining properties of Ionescu-Tulcea Theorem states that for every n

$$\tilde{P}[X_0 \in A_0, \dots, X_n \in A_n] = \Pi_{i=0}^{n} P_i[A_i].$$

Therefore, $(X_n)_{n\geq 0}$ is independent with respect to P if and only if $P = \tilde{P}$.

4. Matlab Files

```
function bmsc24
% In this exercise we simulate 10 smaple paths of a
    drifted BM X= 1+2t+2W_t
tic
%% parameter input
% horizon
```

Siehe nächstes Blatt!

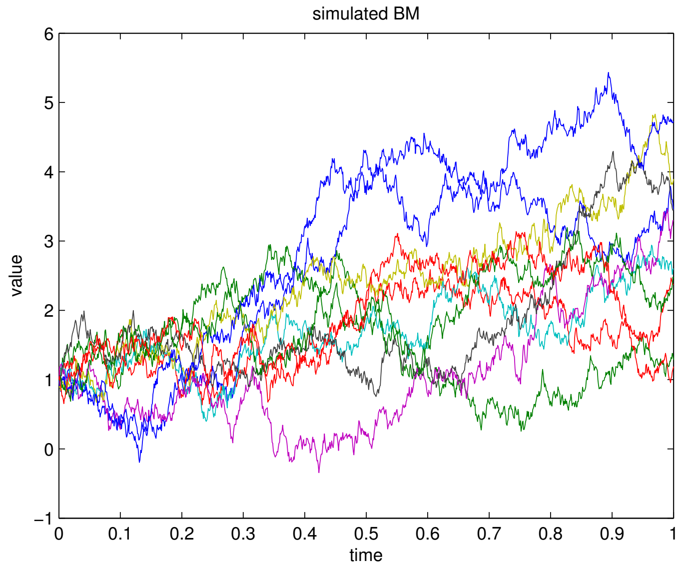

Abbildung 1: 10 sample paths of a standard BM

```
T=1;
% sample size
Nplot=10;
% grid points
M=10^3;
% volatility and drift coeff
sigma=2;
driftcoeff=2;

% Simulate BM with normal increments
BM = [zeros(1,Nplot);sqrt(T/M)*cumsum(randn(M,Nplot))];
% the process X
timegrid= 0:T/M:T;
drift=repmat(timegrid',1,Nplot);
X=1+driftcoeff*drift+sigma*BM;

%plot the sample paths
plot(timegrid,X)
title('simulated BM');
xlabel('time');
ylabel('value');
toc
```