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Brownian Motion and Stochastic Calculus
Sketch of Solution Sheet 3

1. **a)** Fix any $t \in \mathbb{R}$. Since Brownian motion B is a Gaussian process, we get by definition that X_t is Gaussian distributed. It remains to check its mean and variance:

$$\begin{aligned} \mathrm{E}[X_t] &= 0, \\ \mathrm{Var}(X_t) &= e^{-2t}e^{2t} = 1. \end{aligned}$$

b) Fix any $n \in \mathbb{N}$ and any $t_1, t_2, ..., t_n \geq 0$. It is enough to check that

$$\big(X_{-t_1}, X_{-t_2}, ..., X_{-t_n}\big) \overset{Law}{=} \big(X_{t_1}, X_{t_2}, ..., X_{t_n}\big).$$

From the invariance by time inversion property of Brownian motion (cf. Proposition 1.1 in Section 2.1)), we get that for any $\tilde{t}_1, ..., \tilde{t}_n \geq 0$

$$\big(\tilde{t}_1 B_{1/\tilde{t}_1}, \tilde{t}_2 B_{1/\tilde{t}_2}, ..., \tilde{t}_n B_{1/\tilde{t}_n}\big) \overset{Law}{=} \big(B_{\tilde{t}_1}, B_{\tilde{t}_2}, ..., B_{\tilde{t}_n}\big).$$

Therefore, for $\tilde{t}_i := e^{-2t_i}$, $i := 1, ..., n$, we get that

$$\begin{aligned} \big(X_{-t_1}, X_{-t_2}, ..., X_{-t_n}\big) &= \big(e^{t_1} B_{e^{-2t_1}}, e^{t_2} B_{e^{-2t_2}}, ..., e^{t_n} B_{e^{-2t_n}}\big) \\ &\overset{Law}{=} \big(e^{-t_1} B_{e^{2t_1}}, e^{-t_2} B_{e^{2t_2}}, ..., e^{-t_n} B_{e^{2t_n}}\big) \\ &= \big(X_{t_1}, X_{t_2}, ..., X_{t_n}\big). \end{aligned}$$

2. 1) $\Rightarrow$ 4) This follows directly from the uniform integrability of the family $\big\{E[Y|\mathcal{G}] \,\big|\, \mathcal{G} \subseteq \mathcal{F}\big\}$.

4) $\Rightarrow$ 2) As $(X_t)_{t\geq 0}$ is uniformly integrable, it is bounded in $\mathcal{L}^1$, i.e. $\sup_{t\geq 0} E[|X_t|] < \infty$. Applying the supermartingale convergence theorem, we obtain that $X_\infty := \lim_{t\to\infty} X_t$ exists a.s., and $X_\infty \in \mathcal{L}^1$ by Fatou's lemma. By definition, as $(X_t)_{t\geq 0}$ is adapted, X_∞ is $\mathcal{F}_\infty$-measurable. Moreover, by the uniform integrability of $(X_t)_{t\geq 0}$, it converges also in $\mathcal{L}^1$ to X_∞.

2) $\Rightarrow$ 3) From $X_t \to X_\infty$ in $\mathcal{L}^1$, we conclude that also for any $t \geq 0$ and $A \in \mathcal{F}_t$ we have

$$X_t \mathbf{1}_A \to X_\infty \mathbf{1}_A \quad \text{in } \mathcal{L}^1 \text{ for } t \to \infty.$$

Therefore, as $(X_t)_{t\geq 0}$ is a martingale, we have for any $t \geq 0$ and $A \in \mathcal{F}_t$ that

$$E[X_\infty \mathbf{1}_A] = \lim_{s\to\infty} E[X_{t+s} \mathbf{1}_A] = \lim_{s\to\infty} E[X_t \mathbf{1}_A] = E[X_t \mathbf{1}_A],$$

which implies that $E[X_\infty|\mathcal{F}_t] = X_t$, and hence $(X_t)_{t\in[0,\infty]}$ is a martingale.

3) $\Rightarrow$ 1) This is clear for $Y := X_\infty$.

Finally, if 1)–4) hold true, then we have like in the proof of 2) $\Rightarrow$ 3) that for any $t \geq 0$ and $A \in \mathcal{F}_t$,

$$E[X_\infty \mathbf{1}_A] = E[X_t \mathbf{1}_A] = E[Y \mathbf{1}_A], \tag{1}$$

as $X_t = E[Y|\mathcal{F}_t]$. Therefore, (1) holds true for $A \in \bigcup_{t\geq 0} \mathcal{F}_t$. The collection of sets

$$\mathcal{D} := \left\{ A \in \mathcal{F} \,\middle|\, E[X_\infty \mathbf{1}_A] = E[Y \mathbf{1}_A] \right\}$$

is a Dynkin system and contains $\bigcup_{t\geq 0} \mathcal{F}_t$, which is closed under finite intersections. By the Dynkin system theorem, we conclude that $\mathcal{D}$ contains $\mathcal{F}_\infty$ and thus $X_\infty = E[Y|\mathcal{F}_\infty]$.

3. We begin with the first equality: $\sigma(C(S)) = \mathcal{B}(S)$. If h is continuous, then $\{h > c\}$ is open for all $c \in \mathbb{R}$, therefore h is $\mathcal{B}(S)$ measurable and we have $\subseteq$. Conversely, let $A \subseteq S$ be closed, then $h(s) := min(1, d(s, A))$ is in $C(S)$ and $A = \{h = 0\}$, therefore $A \in \sigma(C(S))$ and we have $\supseteq$.
For the second claim notice that for every $t \in [0, 1]$ the map $x \mapsto x(t)$ is a continuous map from S to $\mathbb{R}$. Therefore, $\sigma(\mathcal{Z}) \subseteq \sigma(C(S)) = \mathcal{B}(S)$.
Conversely, since the metric space S is separable, every open set in S can be written as a countable union of balls and because

$$U_\delta(x) = \bigcup_{n\in\mathbb{N}} \overline{U_{\delta-1/n}(x)}, \quad \text{with} \quad U_\varepsilon(x) := \{y \in S | d(y, x) < \varepsilon\},$$

it suffices to show that every closed ε- ball is in $\sigma(\mathcal{Z})$. Indeed,

$$\begin{aligned}\overline{U_\varepsilon(x)} &:= \{y \in S | d(y, x) \leq \varepsilon\} \\ &= \bigcap_{n\in\mathbb{N}} \{y \in S | |y(i/n) - x(i/n)| \leq \varepsilon, i = 0, 1, \ldots, n\} \in \sigma(\mathcal{Z}).\end{aligned}$$

Therefore, $\mathcal{B}(S) \subseteq \sigma(\mathcal{Z})$. Finally, if X is measurable, then for every $t_i \in [0, 1]$ we have $X_{t_i} = F_{t_i} \circ X$ with $F_{t_i} : (S, \mathcal{B}(S)) \mapsto (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ with $f \mapsto f(t_i)$. We see that

Siehe nächstes Blatt!

X_{t_i} is measurable, because X is measurable and F_{t_i} is continuous. Conversely, if all X_{t_i} are measurable then for $Z \in \mathcal{Z}$ we have

$$\begin{aligned} X^{-1}(Z) &= \{\omega \in \Omega | X_{t_i}(\omega) \in A_i, \quad i = 1, \dots, n\} \\ &= \bigcap_{i=1}^{n} X_{t_i}^{-1}(A_i) \in \mathcal{F}. \end{aligned}$$

Hence, X is measurable.

4. Matlab Files

```
function bmscex34
% In this exercise we simulate Brownian motion using the Wiener-Levy
% Representation (see Corollary I.(5.16) in the lecture notes)

% upper bound on n
nmax=12;
% number of iid normal variables
N=sum(2.^[1:nmax]);
% number of sample paths
M=10;
% final time
T=1;
% number of grid points
gridpoi=2000;
% time grid
grid=0:T/gridpoi:T;
% iid std normal random variables
Y=randn(N,M);
% output matrix (N,M)=(N*1)*(1*M) matrix, initialize for n=0: Y_0*phi_0(t)
out=grid'*randn(1,M);

% use the definition of W^N
for n=1:nmax
    for k=1:(2^n)
        % formula I.(5.8)
    out=out+(schauderba(n,k,grid))'*Y(2^(n-1)+k,:);
    end
end
```

Bitte wenden!

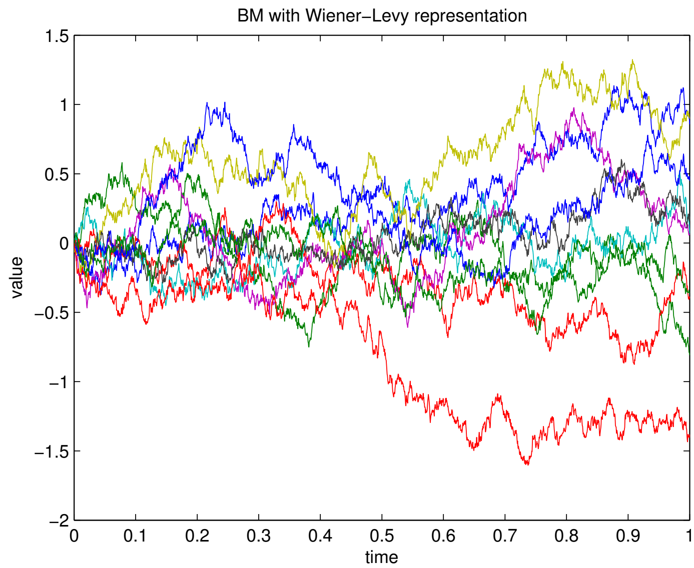

Abbildung 1: 10 sample paths of a BM

```
plot(grid,out)
title('BM with Wiener−Levy representation');
xlabel('time');
ylabel('value');
end


function[value]=schauderba(n,k,t)
% the function implements the schauderbasis function see definition I (5.7)
ind1=t> (2*k−2)*2^(−(n+1));
ind2=t<= (2*k−1)*2^(−(n+1));

ind3=1−ind2;
ind4=t<=2*k*2^(−(n+1));

% Definition of the Schauder basis function definition I (5.7)
value=(ind1.*ind2).*2^(n/2).*(t−(k−1)*2^(−n))...
    −(ind3.*ind4).*2^(n/2).*(t−k*2^(−n));
end
```