

Brownian Motion and Stochastic Calculus

Sketch of Solution Sheet 4

1. a) The density $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ of Z is bounded by $\frac{1}{\sqrt{2\pi}} \leq \frac{1}{2}$. So

$$P[|Z| \leq \varepsilon] = P[-\varepsilon \leq Z \leq \varepsilon] = \int_{-\varepsilon}^{\varepsilon} f(x) dx \leq \frac{1}{2} 2\varepsilon = \varepsilon.$$

Remark: More generally $P[Z \in A] \leq \lambda(A)$, where λ is the Lebesgue measure.

- b) Take any $\alpha > \frac{1}{2}$ and let $M \in \mathbb{N}$ satisfying $M(\alpha - \frac{1}{2}) > 1$. If $W(\omega)$ is locally Hölder-continuous of order α at the point $s \in [0, 1]$, there exists a constant C_h so that $|W_t(\omega) - W_s(\omega)| \leq C_h |t - s|^\alpha$ for t near s . Then $|W_{\frac{k}{n}}(\omega) - W_{\frac{k-1}{n}}(\omega)| \leq \text{const} \cdot n^{-\alpha}$ for all large enough n , for $\frac{k}{n}$ near s and M successive indices k . The set $\{W(\omega) \text{ is locally } \alpha\text{-Hölder at some } s \in [0, 1]\}$ is therefore contained in

$$B := \bigcup_{C \in \mathbb{N}} \bigcup_{m \in \mathbb{N}} \bigcap_{n \geq m} \bigcup_{k=0, \dots, n-M} \bigcap_{j=1}^M \left\{ |W_{\frac{k+j}{n}}(\omega) - W_{\frac{k+j-1}{n}}(\omega)| \leq C \frac{1}{n^\alpha} \right\}.$$

We show that this is a nullset. As the above Brownian increments are iid $\sim N(0, \frac{1}{n})$, we have, with $Z \sim N(0, 1)$, as $P[|Z| \leq \varepsilon] \leq \varepsilon$ for any $\varepsilon \geq 0$ (see a)), that

$$\begin{aligned} P \left[\bigcap_{j=1}^M \left\{ |W_{\frac{k+j}{n}}(\omega) - W_{\frac{k+j-1}{n}}(\omega)| \leq C \frac{1}{n^\alpha} \right\} \right] &= \left(P \left[|Z| \leq \frac{C}{n^{\alpha-1/2}} \right] \right)^M \\ &\leq C^M n^{-M(\alpha-\frac{1}{2})}. \end{aligned} \quad (1)$$

Now, we have

$$\begin{aligned} D_m &:= \bigcap_{n \geq m} \bigcup_{k=0, \dots, n-M} \bigcap_{j=1}^M \left\{ |W_{\frac{k+j}{n}}(\omega) - W_{\frac{k+j-1}{n}}(\omega)| \leq C \frac{1}{n^\alpha} \right\} \\ &\subseteq \bigcup_{k=0, \dots, n-M} \bigcap_{j=1}^M \left\{ |W_{\frac{k+j}{n}}(\omega) - W_{\frac{k+j-1}{n}}(\omega)| \leq C \frac{1}{n^\alpha} \right\} \quad \text{for each } n \geq m \end{aligned}$$

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and therefore, due to (1), as $M(\alpha - \frac{1}{2}) > 1$, we get

$$\begin{aligned} P[D_m] &\leq \limsup_{n \rightarrow \infty} P \left[\bigcup_{k=0, \dots, n-M} \bigcap_{j=1}^M \left\{ |W_{\frac{k+j}{n}}(\omega) - W_{\frac{k+j-1}{n}}(\omega)| \leq C \frac{1}{n^\alpha} \right\} \right] \\ &\leq \limsup_{n \rightarrow \infty} n C^M n^{-M(\alpha - \frac{1}{2})} \\ &= 0. \end{aligned}$$

Therefore, being a countable union of nullsets, B has $P[B] = 0$.

- c) Let $Y_\sigma \sim \mathcal{N}(0, \sigma^2)$ for any $\sigma \geq 0$. We note that $E[Y_\sigma^m] = C_m \sigma^m$, where $C_m = E[Y_1^m]$. Thus

$$E[|W_t - W_s|^{2n}] = C_{2n} |t - s|^n \quad \text{for all } n.$$

Writing $\gamma_n := 2n$ and $\beta_n := n - 1$ yields that

$$E[|W_t - W_s|^{\gamma_n}] = C_{2n} |t - s|^{1+\beta_n} \quad \text{for all } n.$$

Now, fix $\alpha < \frac{1}{2}$. As $\frac{\beta_n}{\gamma_n} < \frac{1}{2}$ for any $n \in \mathbb{N}$ and $\frac{\beta_n}{\gamma_n}$ converges to $\frac{1}{2}$, we find big enough N such that $\alpha < \frac{\beta_N}{\gamma_N}$. Thus, we get the result applying the *Kolmogorov–Čentsov theorem*.

2. a) We know from Proposition (3.4) in section 2.3 of the lecture notes that for any $\lambda \in \mathbb{R}$, the process $M^\lambda := (M_t^\lambda)_{t \geq 0}$ defined by

$$M_t^\lambda = \exp \left(\lambda W_t - \frac{\lambda^2}{2} t \right)$$

is a continuous (P, \mathbb{H}) -martingale. Moreover, for any $n \in \mathbb{N}$, $T_a \wedge n$ is a bounded stopping time. Thus, applying the stopping theorem (see Theorem 3.8 in section 2.3) we get

$$E[M_{T_a \wedge n}^\lambda] = E[M_0^\lambda] = 1.$$

Now, by the law of iterated logarithm for Brownian motion (Theorem (1.3) in section 2.1), we obtain directly that $P[T_a < \infty] = 1$, which proves the first part. Moreover, on the event $\{T_a < \infty\}$ we have

$$\exp \left(\lambda W_{T_a \wedge n} - \frac{\lambda^2}{2} (T_a \wedge n) \right) \xrightarrow{n \rightarrow \infty} \exp \left(\lambda W_{T_a} - \frac{\lambda^2}{2} T_a \right) = e^{\lambda a} \exp \left(-\frac{\lambda^2}{2} T_a \right).$$

We conclude that for any $\lambda > 0$

$$\exp \left(\lambda W_{T_a \wedge n} - \frac{\lambda^2}{2} (T_a \wedge n) \right) \xrightarrow{n \rightarrow \infty} e^{\lambda a} \exp \left(-\frac{\lambda^2}{2} T_a \right) \quad P\text{-a.s.} \quad (2)$$

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Observe that for any $n \in \mathbb{N}$ we have

$$0 \leq \exp \left(\lambda W_{T_a \wedge n} - \frac{\lambda^2}{2} (T_a \wedge n) \right) \leq e^{\lambda a}$$

Thus, we deduce from (2), by applying the dominated convergence theorem, that for any $\lambda > 0$

$$1 = E[M_{T_a \wedge n}^\lambda] \xrightarrow{n \rightarrow \infty} e^{\lambda a} E \left[\exp \left(- \frac{\lambda^2}{2} T_a \right) \right]$$

and so, for any $\lambda > 0$

$$e^{\lambda a} E \left[\exp \left(- \frac{\lambda^2}{2} T_a \right) \right] = 1. \quad (3)$$

Fix any $\mu > 0$. For $\lambda := \sqrt{2\mu}$, (3) yields the desired result.

b) For any $\lambda > 0$, consider the martingale $(N_t^\lambda)_{t \geq 0}$ defined by

$$N_t^\lambda := \frac{M_t^\lambda + M_t^{-\lambda}}{2} = \cosh(\lambda W_t) \exp \left(- \frac{\lambda^2}{2} t \right) = \cosh(\lambda |W_t|) \exp \left(- \frac{\lambda^2}{2} t \right).$$

The procedure in **a)** (using now N^λ instead of M^λ and \bar{T}_a instead of T_a), using the inequality $0 \leq N_{\bar{T}_a \wedge n}^\lambda \leq \cosh(\lambda a)$, yields

$$\cosh(\lambda a) E \left[\exp \left(- \frac{\lambda^2}{2} \bar{T}_a \right) \right] = 1. \quad (4)$$

Fix any $\mu > 0$. For $\lambda := \sqrt{2\mu}$, (4) yields the desired result.

3. a) From the definition of S_t , we get $\frac{1}{t} \log \frac{S_t}{S_0} = \sigma \frac{W_t}{t} + \mu - \frac{1}{2} \sigma^2$. The strong law of large numbers then gives, P -a.s.,

$$\lim_{t \rightarrow \infty} \log \frac{S_t}{S_0} = \begin{cases} +\infty & \text{if } \mu - \frac{1}{2} \sigma^2 > 0, \\ -\infty & \text{if } \mu - \frac{1}{2} \sigma^2 < 0, \end{cases}$$

and therefore, P -a.s.,

$$\lim_{t \rightarrow \infty} S_t = \begin{cases} +\infty & \text{if } \mu - \frac{1}{2} \sigma^2 > 0, \\ 0 & \text{if } \mu - \frac{1}{2} \sigma^2 < 0. \end{cases}$$

b) If $\mu = \frac{1}{2} \sigma^2$, then $S_t = S_0 e^{\sigma W_t}$. From the law of the iterated logarithm we have

$$\limsup_{t \rightarrow \infty} \frac{W_t}{\sqrt{2t \log \log t}} = +1 \text{ } P\text{-a.s.}, \quad \liminf_{t \rightarrow \infty} \frac{W_t}{\sqrt{2t \log \log t}} = -1 \text{ } P\text{-a.s.}$$

As a consequence, P -almost every path $W(\omega)$ oscillates between $+\infty$ and $-\infty$ for $t \rightarrow \infty$. Therefore, S_t oscillates between 0 and $+\infty$ for $t \rightarrow \infty$.

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c) For $\mu = 0$, it is known that S is a martingale. Moreover, by part a), we have that

$$S_t \xrightarrow{t \rightarrow \infty} 0 \quad P\text{-a.s.}$$

As a martingale with $S_0 > 0$, S cannot converge to 0 in L^1 . Thus, S is not uniformly integrable.

4. Matlab Files

```

1 function bmscex44
2 % In this exercise we simulate BM via path refinement
3 % T= final time
4 T = 1;
5 % number of refinements
6 % (you might understand the code better if you set L=1,
      i.e., only one refinement step)
7 L= 8;
8 %N0=number of grid points on the first level
9 N0 =10;
10 %N= number of grid points on the final level
11 N = N0*2^L;
12 % Brownian motion at each refinement step (inititalize)
13 EMX = zeros(L+1, N+1);
14 % timestep
15 h= T/N;
16 % path of Brownian motion at first level cf. Ex2-4
17 B = [zeros(1,1);sqrt(T/N0)*cumsum(randn(N0,1))];
18 EMX(1, 1:2^L:end) = B;
19
20 % time on the last level
21 t = (T*(0:N)'/N);
22 % plot first picture
23
24 subplot(3,3,1); plot((T*(0:N0)'/N0), B);
25
26 % refine the brownian path
27 for i=2:L+1
28     p = 2^(L-i+1);
29     EMX(i,:) = EMX(i-1,:);
30     % new step size
31     h= T/(N0*2^(i-1));
32     % number of new random variables needed
33     len=N/p/2;

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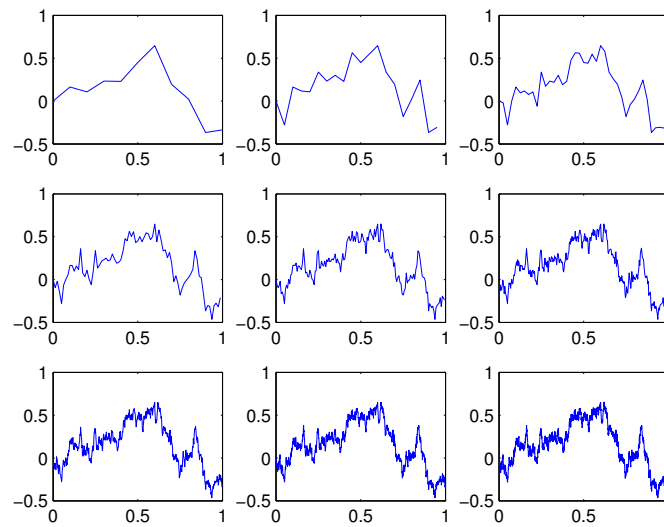


Abbildung 1: refined BM path

```

34      % simulate the BM value at midpoint (t_i+t_(i+1))
      % /2 given the values
35      % at t_i and t_(i+1)
36      % Since BM is a gaussian process , we have that for
      % u<s<t with
37      % u-s=t-s=h the random variable B_s | (B_u=a and
      % B_t=b) is again
38      % gaussian with mean (a+b)/2 and variance h/2
39      EMX(i, [p+1:2*p: N+1-p])= .5*(EMX(i,[1:2*p: N+1-2*p
      ])+EMX(i,[2*p+1:2*p: N+1]))+...
40      sqrt(h./2)*randn(1,len);
41      % plot the refined BM
42      subplot(3,3,i); plot(t(1:p:N), EMX(i, 1:p:N) ');
43  end
44
45 end

```