ETH Zürich FS 2015
D-MATH
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Brownian Motion and Stochastic Calculus
Sketch of Solution Sheet 5

1. To show the reflection principle, let $T_y = \inf\{t > 0 | B_t \geq y\}$ be the first time the BM is greater than y. Then, $\{T_y \leq t\} = \{M_t \geq y\}$ for $y \geq 0$. Furthermore, since $B_{T_y} = y$, we have

$$P(B_t \leq x, M_t \geq y) = P(B_t \leq x, T_y \leq t) = P(B_t - B_{T_y} \leq x - y, T_y \leq t).$$

Relying on the strong Markov property, we obtain

$$\begin{aligned} P(B_t - B_{T_y} \leq x - y, T_y \leq t) &= \mathbb{E}(\mathbf{1}_{\{T_y \leq t\}} P(B_t - B_{T_y} \leq x - y | T_y)) \\ &= \mathbb{E}(\mathbf{1}_{\{T_y \leq t\}} P(B_t - B_{T_y} \leq x - y)) \end{aligned}$$

since $(\tilde{B}_u := B_{T_y+u} - B_{T_y}, u \geq 0)$ is a BM independent of $(B_t, t \leq T_y)$. We also note that $-\tilde{B}$ and $\tilde{B}$ have the same law. Hence,

$$\begin{aligned} \mathbb{E}(\mathbf{1}_{\{T_y \leq t\}} P(B_t - B_{T_y} \leq x - y)) &= \mathbb{E}(\mathbf{1}_{\{T_y \leq t\}} P(B_t - B_{T_y} \geq y - x)) \\ &= \mathbb{E}(\mathbf{1}_{\{T_y \leq t\}} P(B_t - B_{T_y} \geq y - x | T_y)) \\ &= P(B_t \geq 2y - x, T_y \leq t). \end{aligned} \tag{1}$$

The right hand side of (1) is equal to $P(B_t \geq 2y - x)$ since, from $x \leq y$ we have $2y - x \geq y$ which implies that, on the set $\{B_t \geq 2y - x\}$, one has $M_t \geq y$. Therefore, it follows that, for $y \geq 0, x \leq y$,

$$\begin{aligned} P(B_t \leq x, M_t \leq y) &= P(B_t \leq x) - P(B_t \leq x, M_t \geq y) \\ &\overset{(1)}{=} P(B_t \leq x) - P(B_t \geq 2y - x), \end{aligned} \tag{2}$$

and hence the first hint is obtained.
For $0 \leq y \leq x$, since $M_t \geq B_t$ we get

$$P(B_t \leq x, M_t \leq y) = P(B_t \leq y, M_t \leq y) = P(M_t \leq y).$$

Furthermore, by setting $x = y$ in (2)

$$P(B_t \leq y, M_t \leq y) = \Phi\left(\frac{y}{\sqrt{t}}\right) - \Phi\left(-\frac{y}{\sqrt{t}}\right),$$

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hence the second hint is obtained. Finally, noticed that for $y < 0$,

$$P(B_t \leq x, M_t \leq y) = 0$$

since $M_t \geq M_0 = 0$. Finally, the density for the joint law of (B, M) is obtained by taking derivatives.

2. By the first hint we have

$$W_0(S \leq s) = W_0(H_0 \circ \vartheta_s > 1 - s).$$

Let $(\mathcal{H}_t^0)_{t\geq 0}$ be the raw filtration on $C([0,\infty), \mathbb{R})$ generated by (the coordinate process) X. Conditioning on $(\mathcal{H}_s^0)$ and applying the Markov property (Proposition 2.4 in Section 3.2) yields

$$\begin{aligned} W_0(H_0 \circ \vartheta_s > 1 - s) &= E_0[E_0[\mathbf{1}_{\{H_0 > 1-s\}} \circ \vartheta_s | \mathcal{H}_s^0]] \\ &= E_0[W_{X_s}(H_0 > 1 - s)] \end{aligned} \tag{3}$$

Now by symmetry ($-X$ is again a BM), Brownian motion is equally likely to hit 0 starting from x as it is to hit x when starting from 0, i.e.,

$$W_{X_s}(H_0 > 1 - s) = W_0(H_{-X_s} > 1 - s) = W_0(H_{X_s} > 1 - s). \tag{4}$$

Since X is continuous, we have $H_y := \inf\{s > 0 | X_s = y\} = T_y = \inf\{s > 0 | X_s \geq y\}$ for $y > 0$. Moreover, from Ex 5-1 (using the same notation) we know that

$$\begin{aligned} P(T_y \leq t) &= P(M_t \geq y) \\ &= 1 - (\Phi(y/\sqrt{t}) - \Phi(-y/\sqrt{t})). \end{aligned}$$

Therefore, for $y > 0$ the density of T_y is given

$$f_{T_y}(l) = \frac{y}{\sqrt{2\pi l^3}} e^{-y^2/(2l)} \mathbf{1}_{\{l \geq 0\}}. \tag{5}$$

By a complete analogous argument for $y < 0$ we have

$$f_{T_y}(l) = \frac{|y|}{\sqrt{2\pi l^3}} e^{-y^2/(2l)} \mathbf{1}_{\{l \geq 0\}}, \quad \forall y \neq 0. \tag{6}$$

Inserting (6) and (4) back into (3), using Fubini's Theorem and the fact

$$\int x\, e^{-cx^2/2}\, dx = -\frac{e^{-cx^2/2}}{c},$$

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we obtain

$$\begin{aligned}
W_0(S \le s) &= E_0[W_0(H_{X_s} > 1 - s)] \\
&= E_0\Big[\int_{1-s}^{\infty} \frac{|X_s|}{\sqrt{2\pi l^3}}\, e^{-|X_s|^2/(2l)}\, dl\Big] \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi s}}\, e^{-a^2/(2s)} \int_{1-s}^{\infty} \frac{|a|}{\sqrt{2\pi l^3}}\, e^{-a^2/(2l)}\, dl\ da \\
&= 2\int_{0}^{\infty} \frac{1}{\sqrt{2\pi s}}\, e^{-a^2/(2s)} \int_{1-s}^{\infty} \frac{a}{\sqrt{2\pi l^3}}\, e^{-a^2/(2l)}\, dl\ da \\
&= 2\int_{1-s}^{\infty} \frac{1}{2\pi\,\sqrt{sl^3}}\, \Big(\int_{0}^{\infty} a\, e^{-\frac{a^2}{2}(\frac{1}{s}+\frac{1}{l})}\, da\Big)\, dl \\
&= 2\int_{1-s}^{\infty} \frac{1}{2\pi\,\sqrt{sl^3}}\, \Big(\frac{1}{s}+\frac{1}{l}\Big)^{-1} dl \\
&= \frac{1}{\pi}\int_{1-s}^{\infty} \sqrt{\frac{s}{l}}\, \frac{1}{s+l}\, dl \\
&\overset{w=\frac{l}{s}}{=} \frac{1}{\pi}\int_{\frac{1}{s}-1}^{\infty} \frac{1}{\sqrt{w}}\, \frac{1}{1+w}\, dw \\
&\overset{v=\frac{1}{1+w}}{=} \frac{1}{\pi}\int_{0}^{s} \frac{dv}{\sqrt{v(1-v)}} \\
&= \frac{2}{\pi} \arcsin\sqrt{s}.
\end{aligned}$$

3. First, we show $\mathbb{F} \subsetneq \overline{\mathbb{F}}$. Let Γ be any non-Borel subset of $\mathbb{R}$ not containing 0. Then $\{B_0 \in \Gamma\} \subset \{B_0 \neq 0\}$. $\{B_0 \neq 0\}$ is a P-nullset in $\mathcal{F}_0$, so $\{B_0 \in \Gamma\} \in \overline{\mathcal{F}}_0$. On the other hand, suppose that $\{B_0 \in \Gamma\} \in \mathcal{F}_0$. Then $\{B_0 \in \Gamma\} = \{B_0 \in \Gamma'\}$ for some Borel subset $\Gamma' \subset \mathbb{R}$. For any $x \in \mathbb{R}$, there is an $\omega \in C[0,\infty)$ such that $\omega(0) = x$. Thus, $\Gamma = \Gamma'$, a contradiction.

Next, we show $\overline{\mathbb{F}} \subsetneq \widetilde{\mathbb{F}}$. The set $\{B_1 = 0\}$ is a P-nullset in $\mathcal{F}_1$, hence it belongs to $\widetilde{\mathcal{F}}_0$. On the other hand, suppose that $\{B_1 = 0\} \in \overline{\mathcal{F}}_0$. Then there are sets $F, G \in \mathcal{F}_0$ such that $F \subset \{B_1 = 0\} \subset G$ and $P[F] = P[G]$. Since $P[B_1 = 0] = 0$, we thus have $P[G] = 0$. Write $G = \{B_0 \in \Gamma\}$ for some Borel set $\Gamma \subset \mathbb{R}$. For any $x \in \mathbb{R}$, there is an $\omega \in C[0,\infty)$ such that $\omega(0) = x$ and $\omega(1) = 0$. Then $\omega \in \{B_1 = 0\} \subset G$, hence $x = \omega(0) \in \Gamma$. Since x was arbitrary, $\Gamma = \mathbb{R}$ and we conclude $G = \{B_0 \in \mathbb{R}\} = \Omega$, contradicting $P[G] = 0$.

4. Matlab Files

```
function bmsc54
```

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```
% In this exercise we numerically compute the
    distribution of the last
% visit time of a BM in point 0.
tic
%% parameter input
% horizon
T=1;
% sample size
N=10^4;
% grid points
M=10^4;

% Simulate BM with normal increments
BM = [zeros(1,N);sqrt(T/M)*cumsum(randn(M,N))];
% the process X
timegrid= 0:T/M:T;
% initialize the last visiting time
lasttime = zeros(1,N);

% rounding precision
precision= 10^2;
% round the BM
BMround= round(BM*precision)/precision;

for i=1:N
    % if BM hits zero
    if sum(BMround(:,i)==0)>0
        ind= BMround(:,i)==0;
        % last time BM visits zero
        lasttime(i)= max(timegrid(ind));
    end
end
% theoretical distribution function
theof = @(x) 2/pi*asin(sqrt(x));
% empirical distribution function
[F,X] = ecdf(lasttime);
plot(X,F,'r-',X,theof(X),'b-');
title('ArcSin law of BM: distribution function of L');
xlabel('time');
ylabel('probabilities');
legend('empirical','theoretical');
toc
```

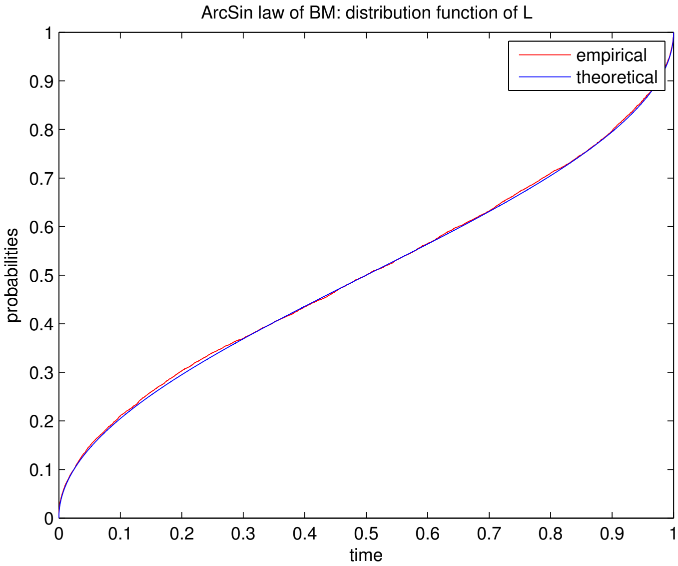

Abbildung 1: theoretical cdf of S (cf. Ex 5-2) vs the empirical cdf