ETH Zürich FS 2015
D-MATH
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# Brownian Motion and Stochastic Calculus
## Sketch of Solution Sheet 6

**1.** For any $g : \mathbb{R} \to \mathbb{R}$ bounded Borel measurable function, we know that

$$E\big[g(|B_t|)\big] = \int_{-\infty}^{\infty} g(|x|) \frac{1}{\sqrt{2\pi t}}\, e^{-x^2/(2t)}\, dx = \int_0^{\infty} g(x) \sqrt{\frac{2}{\pi t}}\, e^{-x^2/(2t)}\, dx \qquad (1)$$

Thus, we see that the probability density function of $|B_t|$ on $\mathbb{R}$ is given by the function

$$x \mapsto \mathbf{1}_{x\geq 0} \sqrt{\frac{2}{\pi t}}\, e^{-x^2/(2t)}.$$

From Corollary 2.55 of the script, we know that the probability density function of the joint law of $(B_t, M_t)$ where $M_t := \sup_{0\leq s\leq t} B_s$ is given by the function

$$(x, y) \mapsto \frac{2(2y-x)}{\sqrt{2\pi t^3}}\, e^{-(2y-x)^2/(2t)}\, \mathbf{1}_{\{y\geq 0,\, x\leq y\}}. \qquad (2)$$

Take any $g : \mathbb{R} \to \mathbb{R}$ bounded Borel measurable function. We deduce from (2) that

$$E\big[g(M_t - B_t)\big] = \int\int_{0\leq y,\, 0\leq y-x} g(y-x)\, \frac{2(2y-x)}{\sqrt{2\pi t^3}}\, e^{-(2y-x)^2/(2t)}\, dx\, dy.$$

By a change of variable $u := y - x$ $v := y$ we get that

$$E\big[g(M_t - B_t)\big] = \int\int_{0\leq u,\, 0\leq v} g(u) \sqrt{\frac{2}{\pi t^3}}\, (u+v)\, e^{-(u+v)^2/(2t)}\, du\, dv \qquad (3)$$

By another change of variable $n := u$ and $m := u + v$ and as $\int x\, e^{-cx^2/2}\, dx = -\frac{e^{-cx^2/2}}{c}$, we get that

$$\begin{aligned} E\big[g(M_t - B_t)\big] &= \int_0^{\infty} g(n) \sqrt{\frac{2}{\pi t^3}} \int_n^{\infty} m\, e^{-m^2/(2t)}\, dm\, dn \\ &= \int_0^{\infty} g(n) \sqrt{\frac{2}{\pi t}}\, e^{-n^2/(2t)}\, dn. \qquad (4) \end{aligned}$$

**Bitte wenden!**

Comparing (1) with (4) yields that $M_t - B_t \overset{Law}{=} |B_t|$.
Now, from (2), we deduce for any $g : \mathbb{R} \to \mathbb{R}$ bounded Borel measurable function that

$$E\big[g(M_t)\big] = \int\int_{0\le y,\, 0\le y-x} g(y)\, \frac{2(2y-x)}{\sqrt{2\pi t^3}}\, e^{-(2y-x)^2/(2t)}\, dx\, dy.$$

By a change of variable $u := y$ and $v := y - x$ we get that

$$E\big[g(M_t)\big] = \int\int_{0\le u,\, 0\le v} g(u)\sqrt{\frac{2}{\pi t^3}}\,(u+v)\, e^{-(u+v)^2/(2t)}\, du\, dv. \tag{5}$$

Comparing (3) with (5) yields $M_t - B_t \overset{Law}{=} M_t$.

**2.** Fix any $t, h \ge 0$ and $f \in b\mathcal{B}(\mathbb{R})$. The case where $h = 0$ is trivial, therefore, let $h > 0$. From the lecture (cf. Example 2.23 in Section 3.2 in the lecture notes), we know that Brownian motion is a Markov process with transition semigroup given by $R_0\tilde{f}(x) = \tilde{f}(x)$ and

$$R_h\tilde{f}(x) = \frac{1}{\sqrt{2\pi h}}\int_{\mathbb{R}} \tilde{f}(y)\exp\left(-\frac{(y-x)^2}{2h}\right)dy \quad \text{when } h > 0, \quad \tilde{f} \in b\mathcal{B}(\mathbb{R}).$$

Therefore, we get for $\tilde{f}(x) := f(|x|) \in b\mathcal{B}(\mathbb{R})$ that

$$\begin{aligned} E\big[f(X_{t+h})\big|\mathcal{G}_t\big] =& R_h\tilde{f}(B_t) \\ =& \frac{1}{\sqrt{2\pi h}}\int_{\mathbb{R}} \tilde{f}(y)\exp\left(-\frac{(y-B_t)^2}{2h}\right)dy \\ =& \frac{1}{\sqrt{2\pi h}}\int_{[0,\infty)} f(y)\exp\left(-\frac{(y-B_t)^2}{2h}\right)dy \\ &+ \frac{1}{\sqrt{2\pi h}}\int_{(-\infty,0)} f(-y)\exp\left(-\frac{(y-B_t)^2}{2h}\right)dy \end{aligned} \tag{6}$$

By a change of variables and by observing that $\{0\}$ is a null set, we deduce from (6) that

$$\begin{aligned} E\big[f(X_{t+h})\big|\mathcal{G}_t\big] =& \frac{1}{\sqrt{2\pi h}}\int_{[0,\infty)} f(y)\,\exp\left(-\frac{(y-B_t)^2}{2h}\right)dy \\ &+ \frac{1}{\sqrt{2\pi h}}\int_{[0,\infty)} f(y)\,\exp\left(-\frac{(y+B_t)^2}{2h}\right)dy \\ =& \tilde{R}_h f(B_t) \end{aligned} \tag{7}$$

By symmetry of the expression in (7), we see that $E\big[f(X_{t+h})\big|\mathcal{G}_t\big] = \tilde{R}_h f(-B_t)$ and thus

$$E\big[f(X_{t+h})\big|\mathcal{G}_t\big] = \tilde{R}_h f(X_t).$$

**Siehe nächstes Blatt!**

**3.** (a) Let $Z_t := M_t - B_t$ and $Y_t := |B_t|$. With the definition of $D$ we have to check that

$$\sup_{0\le t\le 1} Z_t \overset{law}{=} \sup_{0\le t\le 1} Y_t.$$

Since both $Z$ and $Y$ are continuous processes, it suffices to check that

$$\sup_{t\in[0,1]\cap\mathbb{Q}} Z_t \overset{law}{=} \sup_{t\in[0,1]\cap\mathbb{Q}} Y_t. \tag{8}$$

Let $(t_n)_{n\in\mathbb{N}}$ be a counting sequence in $[0,1]\cap\mathbb{Q}$. By Lévy's Theorem, the processes $Z$ and $Y$ have the same law, and therefore for $n\in\mathbb{N}$ the random variables

$$Z_n := \sup(Z_{t_1}, Z_{t_2}, \dots, Z_{t_n}) \quad Y_n := \sup(Y_{t_1}, Y_{t_2}, \dots, Y_{t_n})$$

have the same law. Since $Z_n$ and $Y_n$ converge monotonically to $\sup_{t\in[0,1]\cap\mathbb{Q}} Z_t$ and $\sup_{t\in[0,1]\cap\mathbb{Q}} Y_t$ we have for all $x\in\mathbb{R}$

$$\begin{aligned}
P\left[\sup_{t\in[0,1]\cap\mathbb{Q}} Z_t \le x\right] =& P\left[\bigcap_{n=0}^{\infty}\{Z_n \le x\}\right] \\
=& \lim_{n\to\infty} P[Z_n \le x] \\
=& \lim_{n\to\infty} P[Y_n \le x] \\
=& P\left[\sup_{t\in[0,1]\cap\mathbb{Q}} Y_t \le x\right],
\end{aligned}$$

which yields (8).

(b) We recall the self-similarity property of Brownian motion, i.e., for $c>0$

$$cB_{t/c^2} \overset{law}{=} B_t.$$

Therefore, for $x>0$

$$\begin{aligned}
P[\sup_{0\le t\le 1} |B_t| \le x] =& P[\sup_{0\le t\le 1} |B_{t/x^2}| \le 1] \\
=& P[\sup_{0\le t\le 1/x^2} |B_t| \le 1] \\
=& P[\bar{T}_1 \ge x^{-2}] \\
=& P[1/\sqrt{\bar{T}_1} \le x].
\end{aligned}$$

(c) Using the identity

$$\sqrt{2/\pi}\int_0^\infty e^{-x^2/(2\sigma^2)}dx = \sigma$$

**Bitte wenden!**

and Tonelli's Theorem we have

$$\begin{aligned} E[D] =& E[\sup_{0\le t\le 1} |B_t|] \\ =& E[1/\sqrt{\bar{T}_1}] \\ =& \sqrt{2/\pi} \int_0^\infty E[e^{-x^2\bar{T}_1/2}] dx. \end{aligned}$$

From Ex 4-2 we know that the Laplace transform of $\bar{T}_1$ is

$$E[e^{-\mu\bar{T}_1}] = 1/cosh(\sqrt{2\mu}), \quad \forall \mu > 0.$$

Putting everything together, we have

$$\begin{aligned} E[D] =& \sqrt{2/\pi} \int_0^\infty \frac{dx}{cosh(x)} \\ =& 2\sqrt{2/\pi} \int_0^\infty \frac{e^x dx}{e^{2x}+1} \\ =& 2\sqrt{2/\pi} \int_1^\infty \frac{dy}{y^2+1} \\ =& 2\sqrt{\frac{2}{\pi}} \cdot \frac{\pi}{4} = \sqrt{\frac{\pi}{2}}. \end{aligned}$$

**4. Matlab Files**

```matlab
function bmsc64
% In this exercise we numerically compute the distribution of the maximum
% of a BM on [0,1]-BM_1, the absolute value of BM at time 1 and the maximum
% of a BM on [0,1]
tic
%% parameter input
% horizon
T=1;
% sample size
N=10^4;
% grid points
M=10^4;

% Simulate BM with normal increments
BM = [zeros(1,N);sqrt(T/M)*cumsum(randn(M,N))];
% initialize max(BM)-BM_1, max(BM) and |BM_1|
```

**Siehe nächstes Blatt!**

```
temp1 = repmat(max(BM),M+1,1)-BM;
X1 = temp1(end,:);
X2 = abs(BM(end,:));
temp3 = max(BM);
X3 = temp3(end,:);

% theoretical distribution function (cf. Ex 5-1)
theof = @(x) 2*normcdf(x)-1;
% empirical distribution function
[Fx1,x1]=ecdf(X1);
[Fx2,x2] = ecdf(X2);
[Fx3,x3]=ecdf(X3);
figure(1)
plot(x1,Fx1,'r-',x1,theof(x1),'b-');
title('distribution of max(BM)-BM');
xlabel('time');
ylabel('probabilities');
legend('ecdf of max(BM)-BM','theoretical');

figure(2)
plot(x2,Fx2,'r-',x2,theof(x2),'b-');
title('distribution of abs(BM)');
xlabel('time');
ylabel('probabilities');
legend('ecdf of abs(BM)','theoretical');

figure(3)
plot(x3,Fx3,'r-',x3,theof(x3),'b-');
title('distribution of max of BM');
xlabel('time');
ylabel('probabilities');
legend('ecdf of max BM','theoretical');
toc
end
```

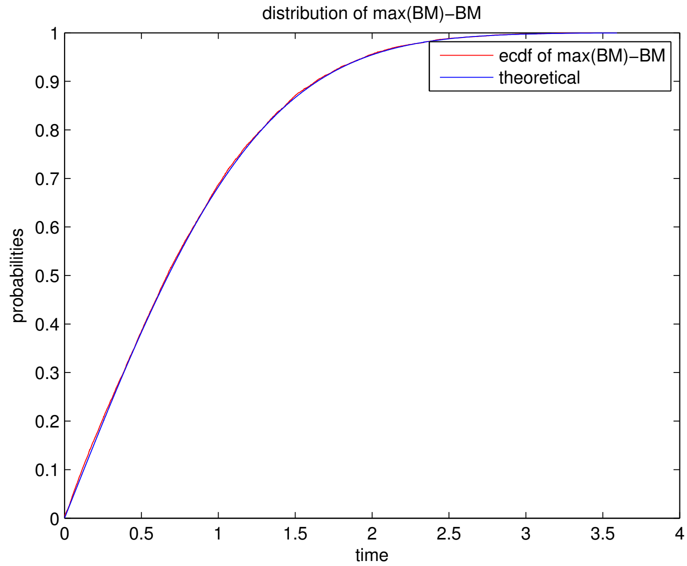


Abbildung 1: theoretical cdf of $M_1 - B_1$ (cf. Ex 6-1) vs the empirical cdf

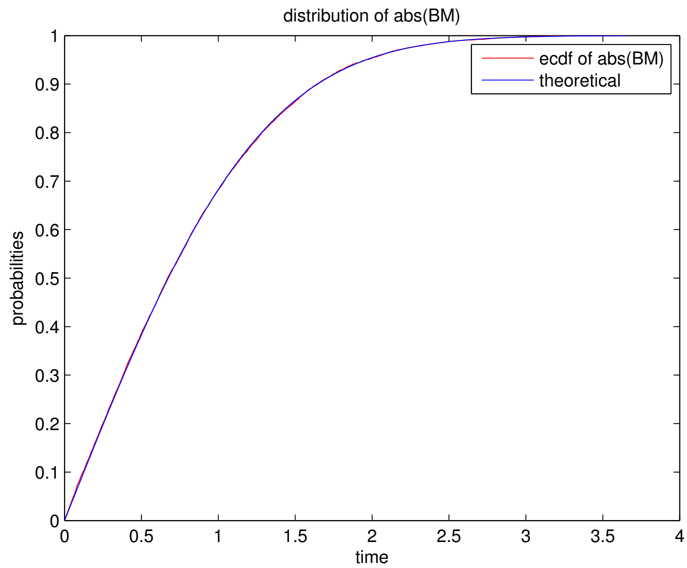


Abbildung 2: theoretical cdf of $|B_1|$ (cf. Ex 6-1) vs the empirical cdf

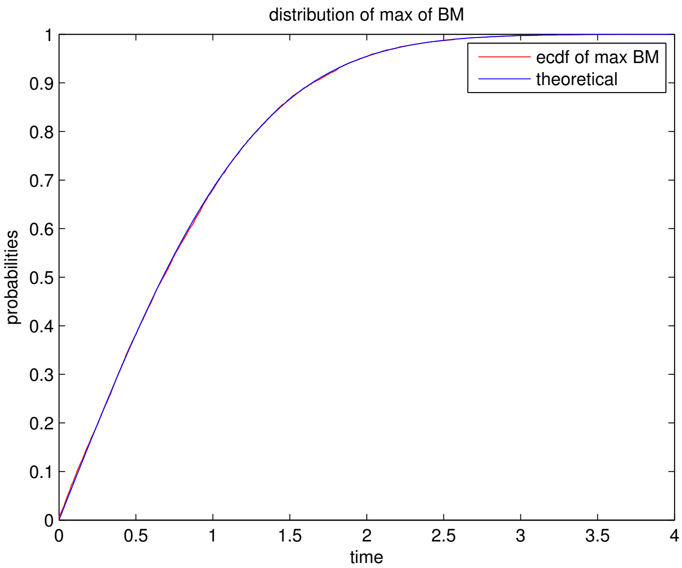


Abbildung 3: theoretical cdf of $M_1$ (cf. Ex 6-1) vs the empirical cdf