ETH Zürich FS 2015
D-MATH
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Brownian Motion and Stochastic Calculus
Sketch of Solution Sheet 7

1. **a)** For the one direction, let $f : [0, \infty) \to \mathbb{R}$ be of finite variation. We define

$$f_1(t) := \frac{|f|(t) + f(t)}{2} \quad \text{and} \quad f_2(t) := \frac{|f|(t) - f(t)}{2},$$

where $|f|(t)$ denotes the variation of f. We claim that both f_1, f_2 are non decreasing. We first show that $|f|$ is non decreasing. For that purpose, fix any $0 \leq s < t$ and denote by Π_s and Π_t the set of finite partition of $[0, s]$ and $[0, t]$, respectively. Moreover, let $\Pi_{s;t}$ be the set of finite partitions of $[0, t]$ such that s is a grid point, i.e. $s = t_i$ for some t_i. Then, we see that

$$|f|(s) \leq \sup_{\Pi_{s;t}} \sum_{t_i \in \Pi} \big|f(t_{i+1}) - f(t_i)\big| \leq \sup_{\Pi_t} \sum_{t_i \in \Pi} \big|f(t_{i+1}) - f(t_i)\big| = |f|(t).$$

So we conclude that $|f|$ is indeed non decreasing. We denote by $\Pi_{[s,t]}$ the set of finite partitions of $[s, t]$. Then, we have

$$\sup_{\Pi_{s;t}} \sum_{t_i \in \Pi} \big|f(t_{i+1}) - f(t_i)\big| = \sup_{\Pi_s} \sum_{t_i \in \Pi} \big|f(t_{i+1}) - f(t_i)\big| + \sup_{\Pi_{[s,t]}} \sum_{t_i \in \Pi} \big|f(t_{i+1}) - f(t_i)\big|.$$

Now, to see that f_1 is non decreasing, it suffices to show that for any $0 \leq s < t$,

$$|f|(t) - |f|(s) \geq -\big(f(t) - f(s)\big).$$

To see this, observe that

$$\begin{aligned} |f|(t) - |f|(s) &\geq \sup_{\Pi_{s;t}} \sum_{t_i \in \Pi} \big|f(t_{i+1}) - f(t_i)\big| - \sup_{\Pi_s} \sum_{t_i \in \Pi} \big|f(t_{i+1}) - f(t_i)\big| \\ &= \sup_{\Pi_{[s,t]}} \sum_{t_i \in \Pi} \big|f(t_{i+1}) - f(t_i)\big| \geq -\big(f(t) - f(s)\big). \end{aligned}$$

To show that f_2 is non decreasing, it suffices to show that

$$|f|(t) - |f|(s) \geq f(t) - f(s),$$

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which we get by applying the same argument as for f_1.
For the other direction, let $f = f_1 - f_2$, where f_1 and f_2 are non decreasing. Then we have for any $t \geq 0$ and any partition Π of $[0, t]$ that

$$\begin{aligned}\sum_{t_i \in \Pi} \big|f(t_{i+1}) - f(t_i)\big| &= \sum_{t_i \in \Pi} \Big|\big(f_1(t_{i+1}) - f_2(t_{i+1})\big) - \big(f_1(t_i) - f_2(t_i)\big)\Big| \\ &\leq \sum_{t_i \in \Pi} \big|f_1(t_{i+1}) - f_1(t_i)\big| + \sum_{t_i \in \Pi} \big|f_2(t_{i+1}) - f_2(t_i)\big| \\ &\leq f_1(t) - f_1(0) + f_2(t) - f_2(0).\end{aligned}$$

Thus, we see that for every $t \geq 0$

$$|f|(t) = \sup_{\Pi} \sum_{t_i \in \Pi} \big|f(t_{i+1}) - f(t_i)\big| < \infty.$$

Remark: f_1, f_2 are not unique.

b) Define $f_1 = |f|$ and $f_2 := |f| - f$. We deduce from **a)** that they are both non decreasing and of course $f = f_1 - f_2$. Moreover, due to the hint, both f_1 and f_2 are continuous. Let μ_{f_1} and μ_{f_2} be the corresponding measures defined in the exercise. Since $\mu_{f_2}\big([0, t]\big) = f_2(t) \leq 2|f|(t) = 2\mu_{|f|}([0, t])$, for every $t \geq 0$, we conclude that $\mu_{f_2} \leq 2\mu_{|f|}$ on $\mathcal{B}\big([0, \infty)\big)$. Thus, we see that

$$\int_0^\infty |g(s)|\, \mu_{f_1}(ds) < \infty, \quad \int_0^\infty |g(s)|\, \mu_{f_2}(ds) < \infty.$$

Therefore,

$$\int_0^\infty g(s)\, \mu_{f_1}(ds) - \int_0^\infty g(s)\, \mu_{f_2}(ds)$$

is well-defined. To show that $\int g\, df$ is well-defined, it remains to show that the definition

$$\int g(s)\, df(s) := \int g(s)\, \mu_{f_1}(ds) - \int g(s)\, \mu_{f_2}(ds)$$

is independent of the choice of f_1, f_2. Let $\overline{f}_1, \overline{f}_2$ be two other functions with the desired properties. Fix any $t \geq 0$. We have for $g(s) := \mathbf{1}_{[0,t]}(s)$ that for any $u \geq 0$

$$\begin{aligned}\int_0^u g(s)\, \mu_{\overline{f}_1}(ds) - \int_0^u g(s)\, \mu_{\overline{f}_2}(ds) &= \overline{f}_1(t \wedge u) - \overline{f}_1(0) - \overline{f}_2(t \wedge u) + \overline{f}_2(0) \\ &= f(t \wedge u) - f(0) \\ &= f_1(t \wedge u) - f_2(t \wedge u) + f_2(0) \\ &= \int_0^u g(s)\, \mu_{f_1}(ds) - \int_0^u g(s)\, \mu_{f_2}(ds).\end{aligned}$$

Thus, we conclude that the same holds true for $g(s) := \mathbf{1}_A$, where $A \in \mathcal{B}(\mathbb{R}_+)$. We conclude that the same holds true for general g Borel by measure theoretical induction.

Siehe nächstes Blatt!

2. **a)** For $M \in \mathcal{M}^c_{0,\text{loc}}$, let $(\tau_n)_{n\in\mathbb{N}}$ be a localizing sequence of stopping times for M such that $(M^2 - \langle M\rangle)^{\tau_n} \in \mathcal{M}^c_0$.

Without loss of generality, we can assume that M^{τ_n} is bounded (otherwise, consider the stopping time $\sigma_n := \inf\left\{t > 0 \middle| |M_t| > n\right\}$ and take $\tau'_n := \tau_n \wedge \sigma_n$ as localizing sequence). We then have that

$$E[M^2_{\tau_n\wedge t}] = E[\langle M\rangle_{\tau_n\wedge t}]. \tag{1}$$

We show the first implication. Let $M \in \mathcal{H}^{2,c}_0$; then by the martingale convergence theorem, $M_t \to M_\infty \in L^2$ a.s. and in L^2. Moreover, Doob's inequality gives $M^*_\infty = \sup_t |M_t| \in L^2$. We can therefore apply the dominated convergence theorem on the left-hand side of (1) for $n \to \infty$.On the other hand, $\langle M\rangle$ is increasing, so that the monotone convergence theorem applied on the right-hand side of (1) for $n \to \infty$ gives

$$E[M^2_t] = E[\langle M\rangle_t] \quad \forall\, t \geq 0.$$

By applying again dominated/monotone convergence theorem for $t \to \infty$, we finally obtain that

$$E[\langle M\rangle_\infty] = E[M^2_\infty] < \infty.$$

Conversely, assume that $E[\langle M\rangle_\infty] < \infty$. By passing to the limit in (1) with the monotone convergence theorem, we obtain that

$$E[M^2_{\tau_n\wedge t}] = E[\langle M\rangle_{\tau_n\wedge t}] \leq E[\langle M\rangle_\infty] =: K < \infty. \tag{2}$$

Therefore, $(M_{\tau_n\wedge t})_{n\in\mathbb{N}}$ is bounded in L^2 and hence uniformly integrable. We can thus pass to the limit in the equality

$$E[M_{\tau_n\wedge t}|\mathcal{F}_s] = M_{\tau_n\wedge s} \quad \forall\, n \in \mathbb{N},$$

obtaining that $E[M_t|\mathcal{F}_s] = M_s$, so that M is a martingale. Moreover, by applying Fatou's lemma to (2), we obtain that

$$E[M^2_t] \leq \liminf_{n\to\infty} E[M^2_{\tau_n\wedge t}] \leq K < \infty,$$

so that M is bounded in L^2, which finally yields that $M \in \mathcal{H}^{2,c}_0$.

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b) Let M be a true martingale and fix $a > 0$. Then the stopping theorem implies

$$M_\tau = E[M_a|\mathcal{F}_\tau],$$

for all stopping times τ with $\tau \leq a$ P-a.s. Since $M_a \in L^1$, we then obtain that $\mathfrak{X}_a$ is uniformly integrable.

Conversely, assume we have a local martingale M of the class (DL), and let $(\tau_n)_{n\in\mathbb{N}}$ be a localizing sequence. Consequently, since $\mathfrak{X}_t$ is uniformly integrable for all $t \geq 0$, we obtain that $\{M_{\tau_n\wedge t}|\ n \in \mathbb{N}\}$ is uniformly integrable. Therefore, the fact that $M_{\tau_n\wedge t} \overset{n\to\infty}{\longrightarrow} M_t$ P-a.s. implies that

$$M_{\tau_n\wedge t} \overset{n\to\infty}{\longrightarrow} M_t \text{ in } L^1.$$

As before, we can therefore pass to the limit in the equality $E[M_{\tau_n\wedge t}|\mathcal{F}_s] = M_{\tau_n\wedge s}$, obtaining that $E[M_t|\mathcal{F}_s] = M_s$, so that M is a martingale.

3. **a)** Since the 3-dimensional Brownian motion $B = (B^1, B^2, B^3)$ takes values in the open set $D := \mathbb{R}^d \setminus \{-x\}$ P-a.s., we can apply Itô's formula to $M_t = f(B_t)$ with $f : D \to (0, \infty)$ given by $f(y) := \frac{1}{|x+y|}$.

For $i = 1, 2, 3$, we have

$$\frac{\partial f}{\partial y^i}(y) = -\frac{x^i + y^i}{|x+y|^3}, \qquad \frac{\partial^2 f}{(\partial y^i)^2}(y) = \frac{-|x+y|^2 + 3(x^i + y^i)^2}{|x+y|^5}.$$

It follows that $\Delta f = \frac{\partial^2 f}{(\partial y^1)^2} + \frac{\partial^2 f}{(\partial y^2)^2} + \frac{\partial^2 f}{(\partial y^3)^2} = 0$ on D. Hence, Itô's formula yields

$$M_t = M_0 + \int_0^t \nabla f(B_s)\, dB_s + \frac{1}{2}\int_0^t \Delta f(B_s)\, ds = \frac{1}{|x|} - \sum_{i=1}^3 \int_0^t \frac{x^i + B_s^i}{|x + B_s|^3}\, dB_s^i.$$

Thus, M is a continuous local martingale.
Let us show the second part. For $t > 0$, using the distribution of the 3-dimensional

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Brownian motion, we obtain that

$$
\begin{aligned}
E\left[|M_t|^2, \mathbf{1}_{\{|M_t| \geq \frac{2}{|x|}\}}\right] &= (2\pi t)^{-\frac{3}{2}} \int\limits_{|x+y| \leq \frac{|x|}{2}} \frac{1}{|x+y|^2} \exp\left(-\frac{|y|^2}{2t}\right) dy \\
&= (2\pi t)^{-\frac{3}{2}} \int\limits_{|y| \leq \frac{|x|}{2}} \frac{1}{|y|^2} \exp\left(-\frac{|y-x|^2}{2t}\right) dy \\
&\leq (2\pi t)^{-\frac{3}{2}} \int\limits_{|y| \leq \frac{|x|}{2}} \frac{1}{|y|^2} \exp\left(-\frac{(|x|-|y|)^2}{2t}\right) dy \\
&\leq (2\pi t)^{-\frac{3}{2}} \exp\left(-\frac{|x|^2}{8t}\right) \int\limits_{|y| \leq \frac{|x|}{2}} \frac{1}{|y|^2} dy \\
&= (2\pi t)^{-\frac{3}{2}} \exp\left(-\frac{|x|^2}{8t}\right) \int_0^{\frac{|x|}{2}} \int_0^{2\pi} \int_0^{\pi} \frac{1}{r^2} r^2 \sin\vartheta \, d\vartheta \, d\phi \, dr \\
&\leq C\,(2\pi t)^{-\frac{3}{2}} \exp\left(-\frac{|x|^2}{8t}\right)
\end{aligned}
$$

where C is a finite positive constant. Now, the function $t \mapsto (2\pi t)^{-\frac{3}{2}} \exp\left(-\frac{|x|^2}{8t}\right)$ is continuous on $(0, \infty)$ and converges to 0 as $t \to 0$ and $t \to \infty$, hence it is bounded on $(0, \infty)$. Therefore, we conclude that

$$
\sup_{t \geq 0} E\left[|M_t|^2\right] \leq \frac{4}{|x|^2} + \sup_{t \geq 0} E\left[|M_t|^2, \mathbf{1}_{\{|M_t| \geq \frac{2}{|x|}\}}\right] < \infty.
$$

It follows that M is bounded in L^2.

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b) For $t > 0$, using spherical coordinates,

$$
\begin{aligned}
E[M_t] &= (2\pi t)^{-3/2} \int_{\mathbb{R}^3} \frac{1}{|x+y|} \exp\left(-\frac{|y|^2}{2t}\right) dy \\
&= (2\pi t)^{-3/2} \int_{\mathbb{R}^3} \frac{1}{|y|} \exp\left(-\frac{|y-x|^2}{2t}\right) dy \\
&\leq (2\pi t)^{-3/2} \int_{\mathbb{R}^3} \frac{1}{|y|} \exp\left(-\frac{(|y|-|x|)^2}{2t}\right) dy \\
&= (2\pi t)^{-3/2} \int_0^\infty \int_0^{2\pi} \int_0^\pi \frac{1}{r} \exp\left(-\frac{(r-|x|)^2}{2t}\right) r^2 \sin\vartheta \, d\vartheta d\phi dr \\
&= 4\pi (2\pi t)^{-3/2} \int_0^\infty r \exp\left(-\frac{(r-|x|)^2}{2t}\right) dr \\
&= 4\pi (2\pi t)^{-3/2} \int_{-|x|}^\infty (r+|x|) \exp\left(-\frac{r^2}{2t}\right) dr \\
&= 4\pi (2\pi t)^{-3/2} \left(\int_{-|x|}^\infty r \exp\left(-\frac{r^2}{2t}\right) dr + |x| \int_{-|x|}^\infty \exp\left(-\frac{r^2}{2t}\right) dr \right) \\
&\leq 4\pi (2\pi t)^{-3/2} \left(\left[-t \exp\left(-\frac{r^2}{2t}\right) \right]_{-|x|}^\infty + |x|\sqrt{2\pi t} \right) \\
&= 4\pi (2\pi t)^{-3/2} \left(t \exp\left(-\frac{|x|^2}{2t}\right) + |x|\sqrt{2\pi t} \right) = O\left(t^{-\frac{1}{2}}\right) \quad (t \to \infty).
\end{aligned}
$$

Hence, $E[M_t] \to 0$ as $t \to \infty$. Since $E[M_0] = \frac{1}{|x|} > 0$, M cannot be a martingale.

4. Matlab Files

```
function prob=bmsc74
% In this exercise we compute the probability of the T_(B_1(0)) being
% finite where T_(B_1(0)) is the first time that a three dimesional BM
% starting at x hits the unit ball B_1(0)
tic

% sample size
N=10^4;
% horizon
T=200;
% grid points
dt=10^(-2);
```

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```
% radius
R=1;
% starting point
x=[1;1;1];

% Simulate BM with normal increments starting at x
BM1 = x(1)+[zeros(1,N);sqrt(dt)*cumsum(randn(T/dt,N))];
BM2 = x(2)+[zeros(1,N);sqrt(dt)*cumsum(randn(T/dt,N))];
BM3 = x(3)+[zeros(1,N);sqrt(dt)*cumsum(randn(T/dt,N))];

% points between [0,T]
m= T/dt+1;

% tau=0, T_(B_1(0))= infty (>T) otherwise 1
tau= zeros(1,N);

for i=1:m
    % compute the norm of BM
    norm=sqrt(BM1(i,:).^2+BM2(i,:).^2+BM3(i,:).^2);
    % is norm(BM) < R?
    ind = norm < R;
    % if yes
    if sum(ind)>0;
        % set tau =1
        tau(ind)=ones(1,sum(ind));
    end
end

% compute the probability
prob= mean(tau);
toc
```